

# Lecture notes .in

## CHAPTER # 10 ROOT LOCUS ANALYSES

After completing this chapter, the students will be able to:

- Describe the root locus,
- State the properties of root locus,
- Sketch the root locus,
- Use root locus to design the gain (K) to meet a transient response specification
- Sketch the root locus for positive feedback systems

### 1. Introduction

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the  $s$  plane as the loop gain is varied.

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans (1950) and used extensively in control engineering. This method, called the *root-locus method*, is one in which the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note



that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.

By using the root-locus method the control Engineer can predict the effects on the location of the closed-loop poles with varying the gain value.

## 2. Root Locus Method (Negative Feedback Systems)

The root locus is the locus of roots of the characteristic equation of the closed-loop system as a specific parameter (usually, gain  $K$ ) is varied from zero to infinity, giving the method its name. Such a plot clearly shows the contributions of each open-loop pole or zero to the locations of the closed-loop poles. Also, by knowing the location of the closed-loop poles, you can discuss the system stability.

Consider the control system shown in Fig. 1

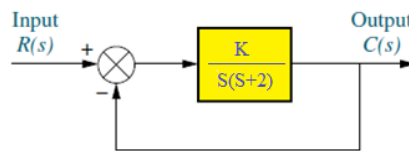


Fig. 1, Control system with unity feedback

The closed-loop T.F. is:

$$\frac{C(s)}{R(s)} = \frac{K}{S^2 + 2S + K}$$

Therefore, the system characteristic eqn. is:

$$S^2 + 2S + K = 0$$

The two poles of the characteristic eqn. change as the gain  $K$  changes from  $0 \rightarrow \infty$  as shown in Fig. 2. The obtained blue line is called root locus.

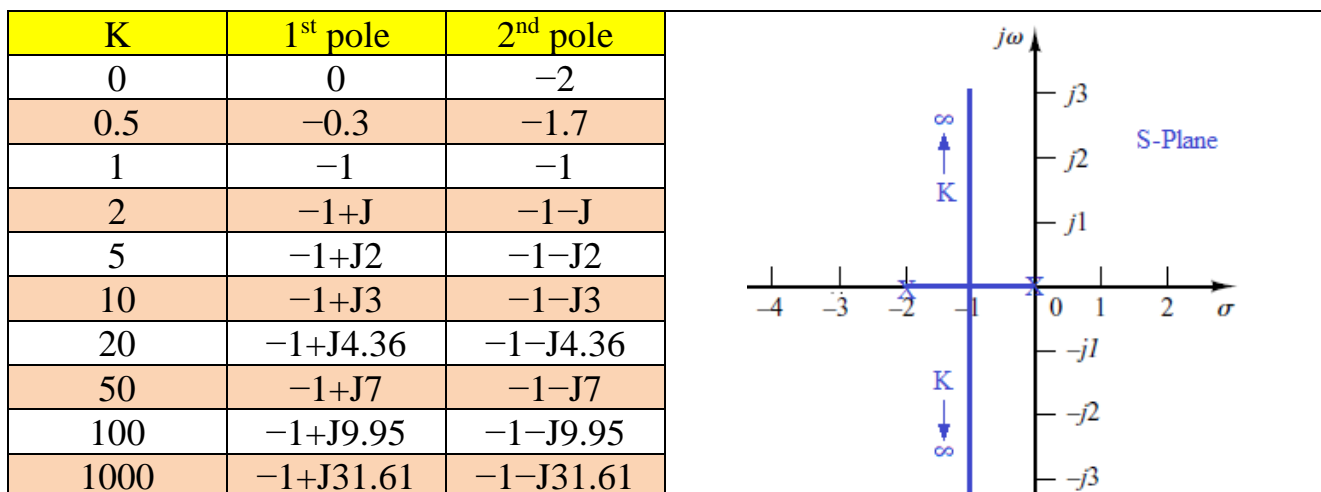


Fig. 2, changing the poles with the gain  $K$



By using the root-locus method, it is possible to determine the value of the gain  $K$  that will make the damping ratio of the dominant closed-loop poles as prescribed.

If the location of an open-loop pole or zero is a system variable, then the root-locus method suggests the way to choose the location of an open-loop pole or zero.

### 2.1 Angle and magnitude conditions

Consider the control system shown in Fig. 3, assuming that the forward T.F include a gain  $K$ , then the closed loop T.F. is;

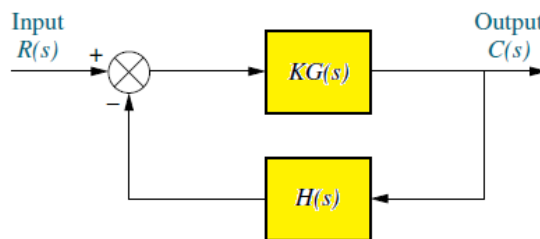


Fig.3 closed loop control system

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

The characteristic equation of this system is;

$$1 + KG(s)H(s) = 0 \rightarrow KG(s)H(s) = -1$$

The quantity  $G(s)H(s)$  is called loop T.F. or open-loop T.F.

$$1 + \frac{K(S + Z_1)(S + Z_2) \dots (S + Z_m)}{S^n(S + P_1)(S + P_2) \dots (S + P_n)} = 0$$

Where  $m$  is the number of zeros and  $n$  is the number of poles.

Since  $KG(s)H(s)$  is a complex quantity, it can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following conditions:

#### Angle condition :

$$\theta = \sum \text{angle of zeros} - \sum \text{angle of poles} = \pm 180$$

$$\theta = \sum_{i=1}^{i=m} \angle(s + Z_i) - \sum_{j=1}^{j=n} \angle(s + P_j) = \pm 180$$



Where a zero angle  $\angle(s + Z_i)$  is the angle measured from +ve real axis of a vector drawn from the zero located at  $Z_i$  to the point  $s$  at which the unknown angle is calculated.

Also, a pole angle  $\angle(s + P_j)$  is the angle measured from +ve real axis of a vector drawn from the pole located at  $P_j$  to the point  $s$  at which the unknown angle is calculated.

For all points located on the root locus, the angle condition is always satisfied.

**Magnitude condition :**

$$|KG(s)H(s)| = 1$$

$$|KG(s)H(s)| = \frac{\prod_{i=1}^m |s + Z_i|}{\prod_{j=1}^n |s + P_j|} = \frac{\text{Product of Zero lengths}}{\text{Product of Pole lengths}} = 1$$

We can calculate the value of gain K using the above relation as:

$$|K| = \frac{1}{|G(s)H(s)|}$$

$$|K| = \frac{\prod_{j=1}^n |s + P_j|}{\prod_{i=1}^m |s + Z_i|} = \frac{\text{Product of Pole lengths}}{\text{Product of Zero lengths}}$$

Where a zero length  $|s + Z_i|$  is the magnitude of vector drawn from the zero located at  $Z_i$  to the point  $s$  at which the gain K is calculated.

Also, a pole length  $|s + P_j|$  is the magnitude of vector drawn from the pole located at  $P_j$  to the point  $s$  at which the gain K is calculated.

For all points located on the root locus, the magnitude condition is always satisfied.

The values of  $s$  that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

**Example (1):**

Consider the T.F. given in Fig. 4,

$$F(s) = \frac{(S + 1)}{S(S + 2)}$$



Find the magnitude and angle of  $F(s)$  at  $S=-3+j4$  and specify if that point lies on the root locus or not.

Sketch the vector of the zero  $(S+1)$  with magnitude  $\sqrt{20}$  and angle  $116.5651^\circ$

Sketch the vector of the pole  $(S)$  with magnitude 5 and angle  $126.8699^\circ$

Sketch the vector of the pole  $(S+2)$  with magnitude  $\sqrt{17}$  and angle  $104.0362^\circ$  as shown below.

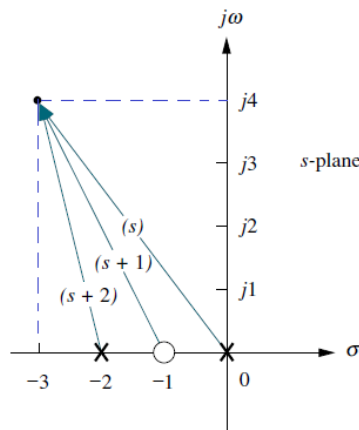


Fig. 4, Vector representation of  $F(s)$

From the angle condition,

$$\theta = 116.5651^\circ - (126.8699^\circ + 104.0362^\circ) = -114.341^\circ$$

Since  $\theta \neq 180$ , therefore at  $S=-3+j4$  is not located on root locus of that system.

From magnitude condition,

$$|KG(s)H(s)| = \frac{\sqrt{20}}{5 \times \sqrt{17}} = 0.2169$$

Since  $|KG(s)H(s)| \neq 1$ , therefore at  $S=-3+j4$  is not located on root locus.

## 2.2 Root Locus Sketch

The root locus of a control system is the locus of the closed-loop poles as the gain  $K$  is varied from zero to infinity.

To begin sketching the root locus of a control system we must know the basic properties of the obtained root locus figure;

### Starting and ending points

Root locus begins at the poles of  $G(s)H(s)$  and ends at the zeros of  $G(s)H(s)$ .



But from the above definition, we draw the root locus as the gain  $K$  is changing from zero to infinity. Therefore,  $K=0$  at the system poles and  $K=\infty$  at the system zeros.

### Number of branches

The number of branches of the root locus equals the number of poles of  $G(s)H(s)$ .

### Symmetry

Due to the open-loop poles and zeros are always symmetrical around the real axis, the obtained root locus is also symmetrical around this axis. Therefore, we only need to construct the upper half of the root locus, then we draw the mirror image of the upper half in the lower half of  $S$ -plane. Fig. 5 explain the above properties.

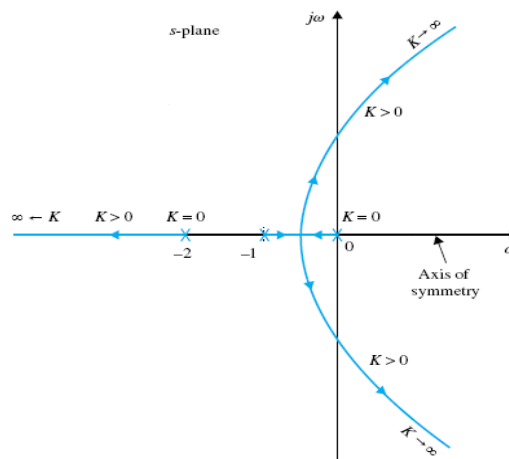


Fig. 5, Properties of root locus

To sketch the root locus of a control system, we must follow the 9 steps given below:

- **Step #1 Location of system poles**

**K=0 points** which are the starting points and located at the open-loop poles

- **Step #2 Location of system zeros**

**K=∞ points** which are the ending points and located at the open-loop zeros

The poles and zeros referred above include those at infinity, if any.

### **Example (2):**

Consider the characteristic equation

$$S(S + 2)(S + 3) + K(S + 1) = 0$$

Dividing both sides by the terms that do not contain  $K$ , we get



$$1 + KG(s)H(s) = 1 + \frac{K(S + 1)}{S(S + 2)(S + 3)} = 0$$

$$G(s)H(s) = \frac{K(S + 1)}{S(S + 2)(S + 3)}$$

When  $K = 0$ , the 3 poles are at  $s = 0$ ,  $s = -2$ , and  $s = -3$

$$\left. \begin{array}{l} s = 0 \\ s = -2 \\ s = -3 \end{array} \right\} n = 3$$

When  $K$  is  $\infty$ , the 3 zeros are at  $s = -1$ ,  $s = \infty$  and  $\infty$

$$\left. \begin{array}{l} s = -1 \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 1$$

The open-loop poles and zeros are given in Fig. 6.

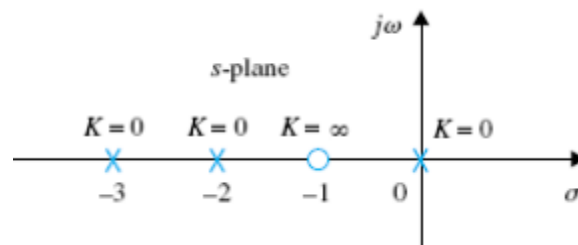


Fig. 6,  $K=0$  and  $K=\infty$  points of the root locus

- **Step #3 Number of Asymptotes**

if there are zeros located at  $\infty$ , there are asymptotes equal to those zeros at  $\infty$ . Simply we can calculate the number of asymptotes by:

$$\text{Number of asymptotes} = |n - m|$$

For the previous example, since there are **TWO** zeros at infinity OR  $n - m = 2$ , there are **TWO** asymptotes.

- **Step #4 Angle of Asymptotes**

we can calculate the angles of asymptotes by

$$\theta_k = \frac{(2k + 1)180}{|n - m|}$$

Where  $k = 0, 1, 2, \dots, |n - m| - 1$

Substituting  $k=0$  we get the angle of 1<sup>st</sup> asymptote

Substituting  $k=1$  we get the angle of 2<sup>st</sup> asymptote, ... etc.



In case of number of asymptotes =2, therefore the angles are  $\theta_0 = 90$  and  $\theta_1 = 270$  (as shown in Fig. 7-a)

In case of number of asymptotes =3, therefore the angles are  $\theta_0 = 60$ ,  $\theta_1 = 180$  and  $\theta_2 = 300$  (as shown in Fig. 7-b)

In case of number of asymptotes =4, therefore the angles are  $\theta_0 = 45$ ,  $\theta_1 = 135$ ,  $\theta_2 = 225$  and  $\theta_3 = 315$  (as shown in Fig. 7-c)

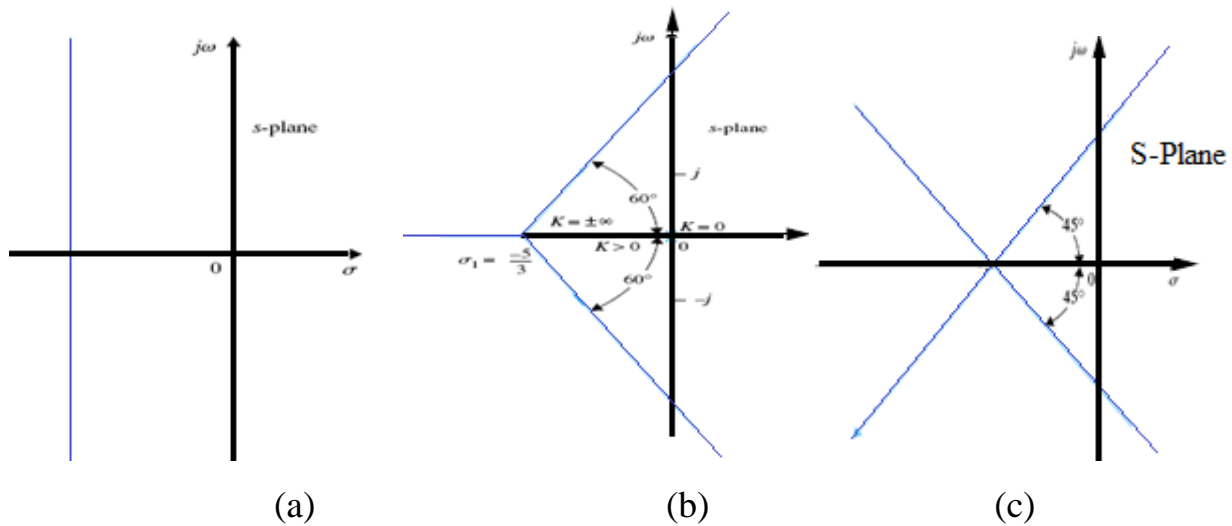


Fig. 7, angle of asymptotes

• **Step #5 Intersection of Asymptotes with Real Axis**

The point of intersection of asymptotes of the root locus lies on the real axis of the s-plane, at  $\sigma$ , where

$$\sigma = \frac{\sum \text{finite poles of } G(s)H(s) - \sum \text{finite zeros of } G(s)H(s)}{|n - m|}$$

The point of intersection of the asymptotes ( $\sigma$ ) represents the center of gravity of the root locus, and is always a real number. Since the poles and zeros of  $G(s)H(s)$  are either real or in complex-conjugate pairs, the imaginary parts in the numerator of  $\sigma$  equation always cancel each other out. Thus, the summation terms may be replaced by the real parts of the poles and zeros of  $G(s)H(s)$ , respectively. That is,

$$\sigma = \frac{\sum \text{real part of poles of } G(s)H(s) - \sum \text{real part of zeros of } G(s)H(s)}{\text{number of asymptotes}}$$





**Example (3):**

suppose we have a control system

$$G(s)H(s) = \frac{K(S + 1)}{S(S + 4)(S^2 + 2S + 2)}$$

$$G(s)H(s) = \frac{K(S + 1)}{S(S + 4)(S + 1 - j)(S + 1 + j)}$$

The point of intersection of asymptotes with real axis is

$$\sigma = \frac{(0 - 4 - 1 - 1) - (-1)}{4 - 1} = \frac{-5}{3} = -1.67$$

• **Step #6 Root Locus segments on Real Axis**

On a given section of the real axis, root locus is found in this section only if the total number of poles and zeros of  $G(s)H(s)$  to the right of the section is *odd*. On another explanation, for  $s_1$  to be a point on the root locus, there must be an *odd* number of poles and zeros of  $G(s)H(s)$  to the right of that point. We can explain this by the following pole-zero configurations shown in Fig. 8.

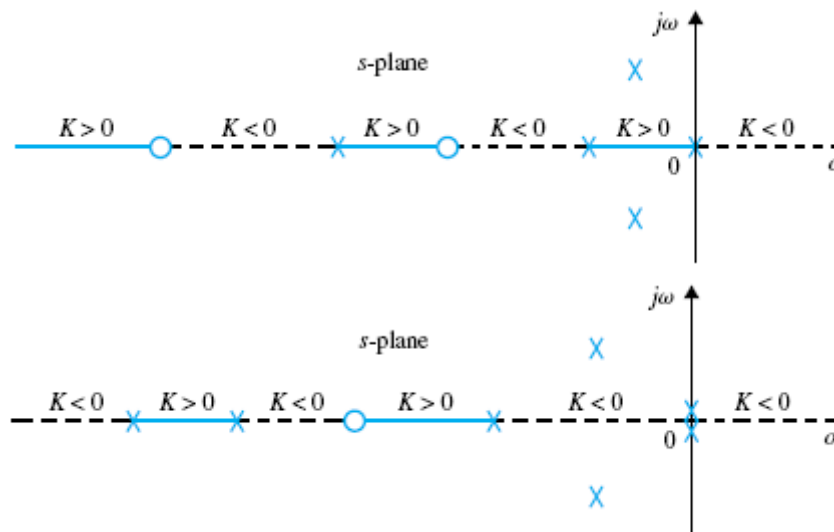


Fig. 8, root locus segments on real axis

Where the dotted line shows Inverse Root Locus (IRL) where the system gain  $K$  changes from  $-\infty$  to 0 (i.e.  $K < 0$ ) used for +ve feedback system (see section 3).

On the other hand, the solid line shows Root Locus (RL) where the system gain  $K$  changes from 0 to  $\infty$  (i.e.  $K > 0$ )



• **Step #7 Angles of Departure for complex poles and/or Angles of Arrival for complex zeros**

The angle of departure or arrival of a root locus at a pole or zero, respectively, of  $G(s)H(s)$  denotes the angle of the tangent to the locus near the point. The angle of departure is defined as the angle at which the root locus leaves the pole. The angle of arrival is defined as the angle at which the root locus moves toward the zero.

**Example (4):**

Consider the characteristic equation of a control system

$$S(S+3)(S^2+2S+2) + K(S+1) = 0$$

The angle of departure of the root locus at  $(s + 1 - j)$  is represented by  $\theta_2$ , measured with respect to the real axis. Let us assign  $s_1$  to be a point on the RL leaving the pole at  $(s + 1 - j)$  and is very close to the pole as shown in Fig. 9. Then,  $s_1$  must satisfy the angle condition. Thus,

$$0 - (\theta_1 + \theta_D + \theta_3 + \theta_4) = 180$$

$$\theta_D = -180 - \theta_1 - \theta_3 - \theta_4 = -180 - 135 - 90 - 26.6 = -71.6$$

Where  $\theta_D$  is the angle of departure of the complex poles.

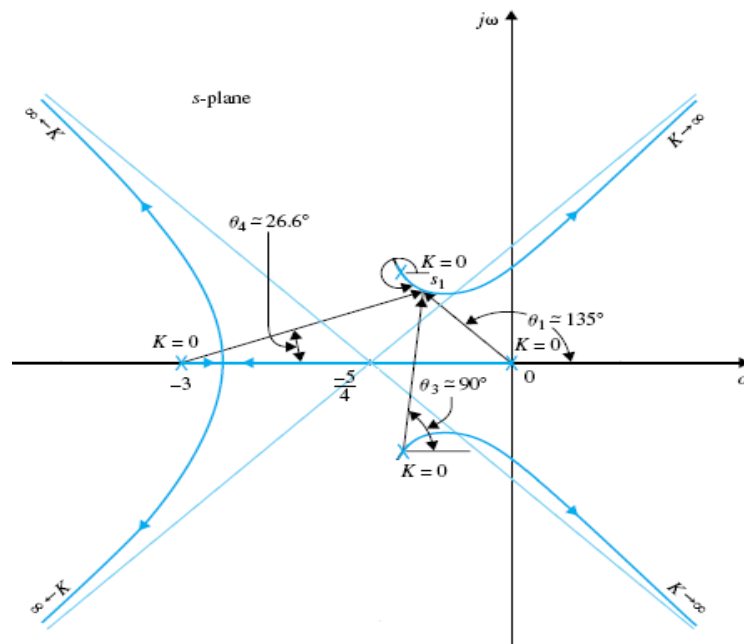


Fig. 9, calculation of angle of departure



**Step #8 Intersection of the Root Locus with the Imaginary Axis**

The points where the root locus intersect the imaginary axis of the  $s$ -plane, and the corresponding values of  $K$ , may be determined by means of the Routh-Hurwitz criterion explained in the previous chapter.

**Step #9 Breakaway Points**

Breakaway points, on the root locus or complementary root locus, are the points at which two or more poles of the characteristic equation are collide.

Fig. 8, illustrates a case in which two branches of the root locus meet at the breakaway point on the real axis and then depart from the axis in opposite directions. In this case, the breakaway point represents a double root of the equation when the value of  $K$  is assigned the value corresponding to the point. Fig. 10, shows another common situation when two complex-conjugate root locus approach the real axis, meet at the break in point, and then depart in opposite directions along the real axis.

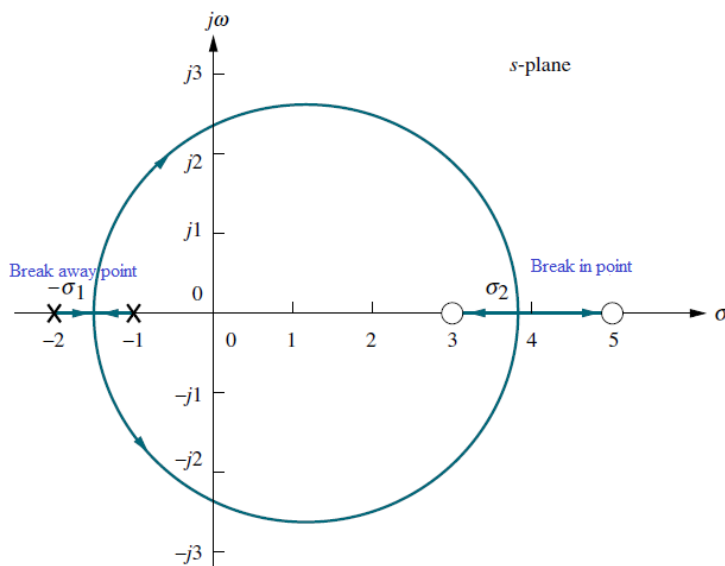


Fig. 10, break away and break in points

A root-locus diagram can have more than one breakaway point. Moreover, the breakaway points need not always be on the real axis. Because of the conjugate symmetry of the root loci, the breakaway points not on the real axis must be in complex conjugate pairs.



To calculate the breakaway points, we must assign  $K=1$  and in that case the open loop T.F is  $G_1(s)H_1(s)$  then obtain the derivative:

$$\frac{dG_1(s)H_1(s)}{ds} = 0$$

*The breakaway point obtained from above eqn. is necessary but not sufficient.*

The angles at which the root locus arrive or depart from a breakaway point depend on the number of branches that are involved at the point. For example, the root locus shown in Fig. 10 all arrive and break away at  $180^\circ$  apart. In general,  $n$  root locus branches arrive or depart a breakaway point at  $360/n$  degrees apart.

**Example (8):**

Consider the characteristic equation of a control system

$$S(S + 2) + K(S + 4) = 0$$

$$G_1(s)H_1(s) = \frac{S + 4}{S(S + 2)}$$

The breakaway points on the root loci must satisfy

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{S(S + 2) - (S + 4) \times (2S + 2)}{S^2(S + 2)^2} = 0$$

$$S(S + 2) - (S + 4) \times (2S + 2) = 0$$

$$S^2 + 8S + 8 = 0$$

We find the two breakaway points of the root locus at  $s = -1.172$  and  $-6.828$

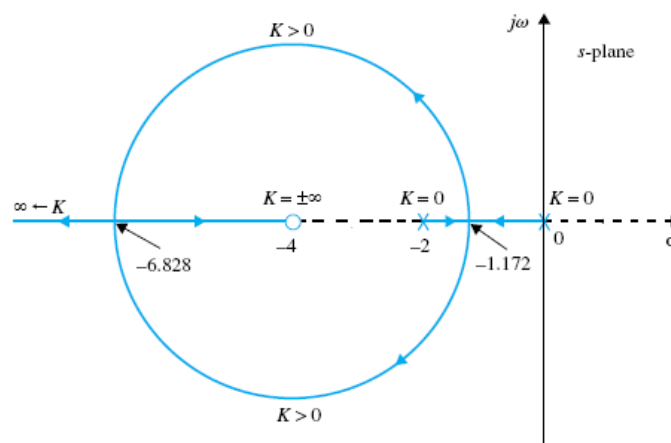


Fig. 11 Breakaway points



Fig. 11, shows that the two breakaway points are all on the root locus for positive  $K$ . this mean both points are used as breakaway points.

**Example (9):**

Consider the characteristic equation

$$S^2 + 2S + 2 + K(S + 2) = 0$$

$$G_1(s)H_1(s) = \frac{(S + 2)}{S^2 + 2S + 2}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{S^2 + 2S + 2 - (S + 2) \times (2S + 2)}{(S^2 + 2S + 2)^2} = 0$$

$$S^2 + 4S + 2 = 0$$

The solution of this equation gives the breakaway point as  $S_1 = -0.586$  and  $S_2 = -3.414$ . The root locus shown in Fig. 12, indicates that, the point  $S_1$  is located in Inverse Root Locus (IRL) portion, this means that point is proper breakaway point for IRL. The other point  $S_2 = -3.414$  is used in plotting root locus.

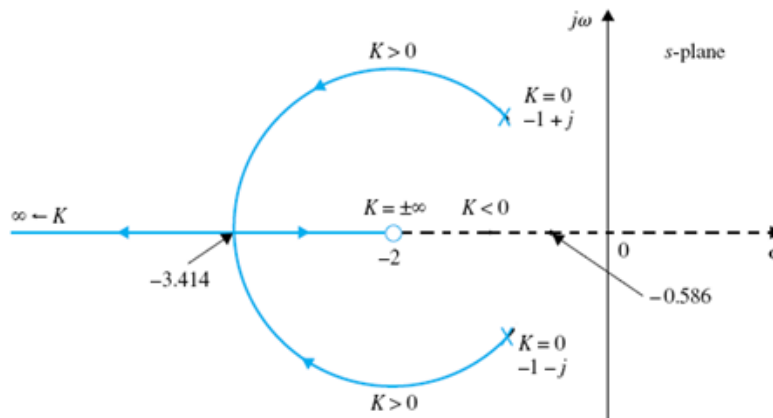


Fig. 12, Breakaway points of the example

**Example (10):**

$$S(S + 4)(S^2 + 4S + 20) + K = 0$$

$$G_1(s)H_1(s) = \frac{1}{S(S + 4)(S^2 + 4S + 20)}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{-(4S^3 + 24S^2 + 72S + 80)}{S^2(S + 4)^2(S^2 + 4S + 20)^2} = 0$$

$$S^2 + 4S + 2 = 0$$

$$4S^3 + 24S^2 + 72S + 80 = 0$$

$$S^3 + 6S^2 + 18S + 20 = 0$$



The solutions of the last equation are  $s_1 = -2$ ,  $s_2 = -2 + j2.45$ , and  $s_3 = -2 - j2.45$ . As shown in Fig. 13, the 3 points are used especially the two complex points  $S_2$  and  $S_3$  are considered as breakaway points.

substitute by  $S_1$  in the characteristic equation and obtain the corresponding value of  $K$ . If  $K$  is real +ve, then the point is accepted as break away point.

$$K = -S(S + 4)(S^2 + 4S + 20)$$

$$K = -\{-2(-2 + 4)((-2)^2 + 4(-2) + 20)\} = 64 \text{ real + ve}$$

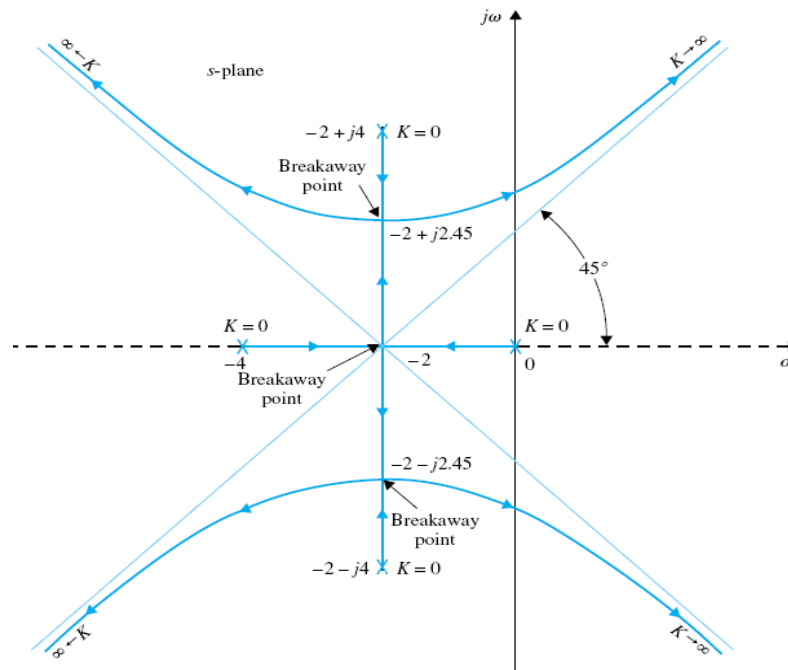


Fig. 13, Breakaway points of the example

**Example (11):**

Consider the root locus of the equation

$$S^3 + 2S^2 + 2S + K = 0$$

$$G_1(s)H_1(s) = \frac{1}{S^3 + 2S^2 + 2S}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{-(3S^2 + 4S + 2)}{(S^3 + 2S^2 + 2S)^2} = 0$$

$$3S^2 + 4S + 2 = 0$$

The roots of the above equation are  $s_1 = -0.667 - j0.471$  and  $s_2 = -0.667 + j0.471$ . From the root locus shown in Fig. 14, these two roots are **not** breakaway points on



the root locus. Moreover, we can check if the point  $S_1$  or  $S_2$  is break away point or not, substitute these values in the characteristic equation and obtain the corresponding value of  $K$ . If  $K$  is real +ve, then the point is accepted as break away point.

$$K = -S^3 - 2S^2 - 2S$$

$$K = -(-0.667 + j0.471)^3 - 2(-0.667 + j0.471)^2 - 2(-0.667 + j0.471)$$

$$K = 0.74074 - j0.2095 \text{ (not real + ve)}$$

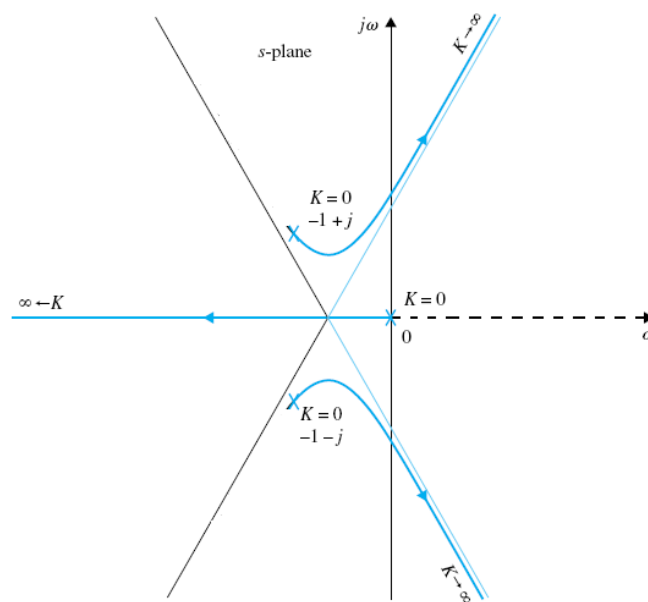


Fig. 14, All Breakaway points are not accepted

### 2.3 Calculation of K from the Root Locus

Once the root locus is constructed, the values of  $K$  at any point (as  $s_1$  for example) on the locus can be determined by using the angle condition described in eqn. (3).

Graphically, the magnitude of  $K$  can be written as:

$$|K| = \frac{\prod_{j=1}^n |s + P_j|}{\prod_{i=1}^m |s + Z_i|} = \frac{\text{Product of Pole lengths}}{\text{Product of Zero lengths}}$$

Note that if there is no finite zeros, the denominator of the above equation is replaced by 1.



$$|K| = \frac{\prod_{j=1}^n |s + P_j|}{\prod_{i=1}^m |s + Z_i|} = \frac{\text{Product of Pole lengths}}{1}$$

For the root locus shown in Fig. 15, the value of  $K$  at point  $s_1$  is given by

$$K = \frac{A \times B}{C}$$

where  $A$  and  $B$  are the lengths of the vectors drawn from the poles of  $G(s)H(s)$  to the point  $s_1$ , and  $C$  is the length of the vector drawn from the zero of  $G(s)H(s)$  to  $s_1$ . In this case,  $s_1$  is on the locus where  $K$  is positive

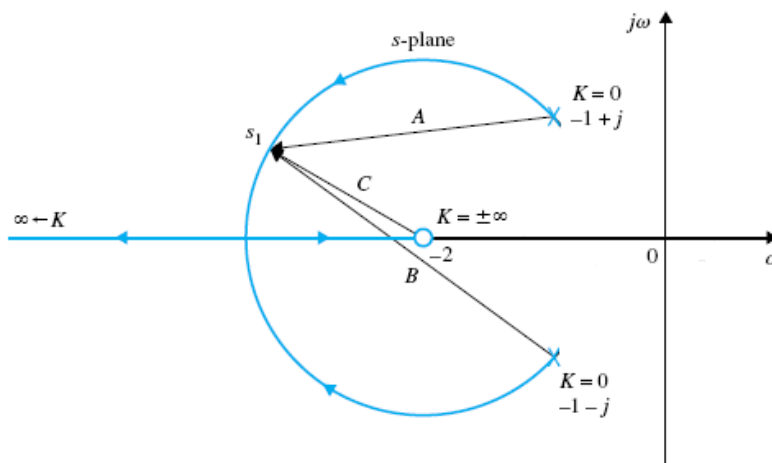


Fig. 15 Calculation of  $K$  using magnitude condition

**Example (12):**

Draw the root locus for the control system whose characteristic equation is

$$S(S + 5)(S + 6)(S^2 + 2S + 2) + K(S + 3) = 0$$

Dividing both sides of the last equation by the terms that do not contain  $K$ ,

$$G(s)H(s) = \frac{K(S + 3)}{S(S + 5)(S + 6)(S^2 + 2S + 2)}$$

- $K = 0$  points: at  $s = 0$ ,  $s = -5$ , and  $s = -6$  and  $S = -1 + j$  and  $S = -1 - j$ .

$$\left. \begin{array}{l} s = 0 \\ s = -5 \\ s = -6 \\ s = -1 - j \\ s = -1 + j \end{array} \right\} n = 5$$

- $K = \infty$  points, at  $s = -3$ .





$$\left. \begin{array}{l} s = -3 \\ s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 1$$

- Number of asymptotes =  $|n - m| = 5 - 1 = 4$
- Angle of asymptotes  $\theta_0 = 45, \theta_1 = 135, \theta_2 = 225$  and  $\theta_3 = 315$
- Intersection of asymptotes with real axis:

$$\sigma = \frac{(-5 - 6 - 1 - 1) - (-3)}{4} = -2.5$$

- Angle of departure: from the angle condition,  

$$\tan^{-1} \frac{1}{2} - \left( 180 - \tan^{-1} \frac{1}{1} - 90 - \tan^{-1} \frac{1}{4} - \tan^{-1} \frac{1}{5} + \theta_D \right) = 180$$

$$26.6 - 135 - 90 - 14.04 - 11.31 - \theta_D = 180 \quad \text{so that } \theta_D = -43.8$$
- Intersection of the root locus with the imaginary axis is determined using Routh's tabulation.

The system characteristic equation is

$$S^5 + 13 S^4 + 54 S^3 + 82 S^2 + (60+K) S + 3 K = 0$$

Using Routh array

$S^5$	1	54	$60+K$
$S^4$	13	82	$3K$
$S^3$	$\frac{620}{13}$	$\frac{780 + 10K}{13}$	
$S^2$	$\frac{3130.7692 - 10K}{47.69231}$	$3K$	
$S^1$	$\frac{-(100K^2 + 65200K - 2441999.976)}{3130.7692 - 10K}$		
$S^0$	$3K$		

From the above array

From  $S^0$  row  $3K > 0 \rightarrow K > 0$  (1)

From  $S^2$  row  $3130.7692 - 10K > 0 \rightarrow K < 313.07692$  (2)

From  $S^1$  row  $-(K^2 + 652K - 24420) > 0 \rightarrow K^2 + 652K - 24420 < 0$   
 $(K - 35.519)(K + 687.519) < 0$



In that case we have 2 scenarios:

First Scenario

$$(K - 35.519) > 0 \rightarrow K > 35.519$$

$$(K + 687.519) < 0 \rightarrow K < -687.519$$

Second Scenario

$$(K - 35.519) < 0 \rightarrow K < 35.519 \quad (3)$$

$$(K + 687.519) > 0 \rightarrow K > -687.519 \quad (4)$$

Due to condition 1 & 2, we accept the Second Scenario

From conditions (1) & (4):  $K > 0$

From conditions (2) & (3):  $K < 35.519$

The range of  $K$  for stability is

$$0 < K < 35.519$$

The maximum value of  $K$  for stability is at  $K = 35.519$

This value is obtained from the row  $S^1$  so the auxiliary equation  $A(S)$  can be obtained from  $S^2$

$$A(s) = \frac{3130.7692 - 10K}{47.69231} S^2 + 3K = 0$$

$$A(s) = 58.19763S^2 + 106.557 = 0$$

$$S = \pm j 1.353$$

- Break away points:

Taking the derivative after making  $K = 1$  we can obtain

$$s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0$$

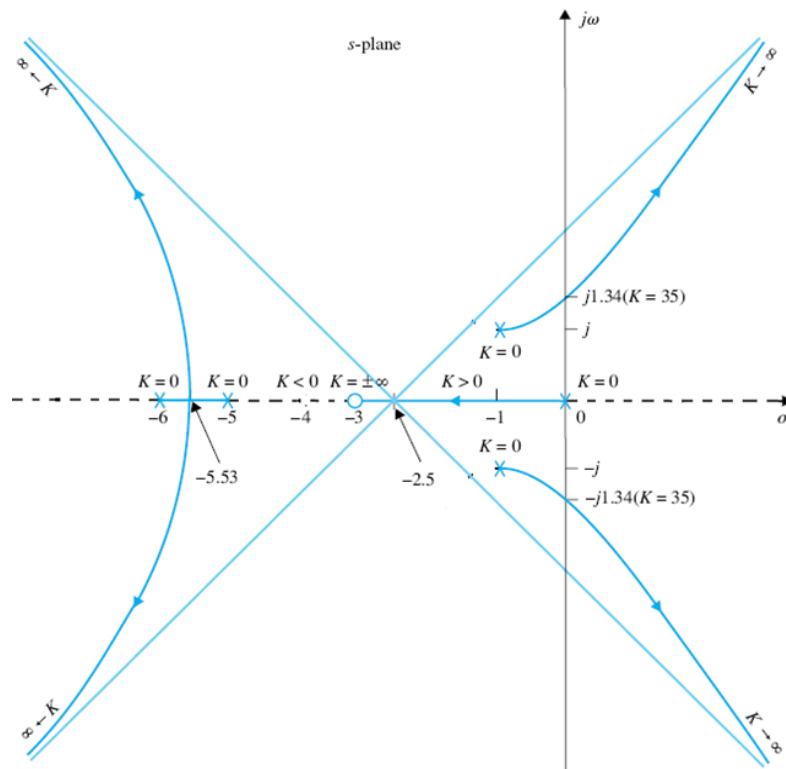
Since there is only one breakaway expected, only one root of the last equation is the correct solution of the breakaway point. The five roots of the above equation are:

$$s = 3.33 + j1.204 \quad s = 3.33 - j1.204$$

$$s = -0.656 + j0.468 \quad s = -0.656 - j0.468$$

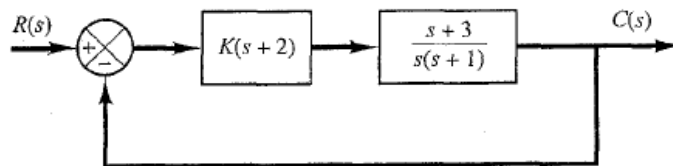
$$s = -5.53$$

Clearly, the breakaway point is at  $-5.53$ . The other four solutions do not satisfy the requirements.



**Example (13):**

For the control system described by the block diagram shown below, draw the root locus.



$$G(s)H(s) = \frac{K(S + 2)(S + 3)}{S(S + 1)}$$

- $K = 0$  points: at  $s = 0$ , and  $s = -1$ .  

$$\left. \begin{matrix} s = 0 \\ s = -1 \end{matrix} \right\} n = 2$$
- $K = \infty$  points, at  $s = -2$  and  $s = -3$ .  

$$\left. \begin{matrix} s = -2 \\ s = -3 \end{matrix} \right\} m = 2$$
- Number of asymptotes =  $|n - m| = 2 - 2 = 0$
- Break away points:



$$G_1(s)H_1(s) = \frac{(S + 2)(S + 3)}{S(S + 1)}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{S(S + 1) \times (2S + 5) - (S + 2)(S + 3) \times (2S + 1)}{S^2(S + 1)^2} = 0$$

$$S(S + 1) \times (2S + 5) - (S + 2)(S + 3) \times (2S + 1) = 0$$

$$S^2 + 3S + 1.5 = 0$$

The roots of the above equation are  $s_1 = -0.634$  and  $s_2 = -2.366$

(Because point  $s = -0.634$  lies between two poles, it is a breakaway point, and because point  $s = -2.366$  lies between two zeros, it is a break-in point.)

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point  $s = -0.634$ , the value of  $K$  is

$$K = - \frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at  $s = -2.366$ ,

$$K = - \frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

The root locus of this example is shown in Fig. 16.

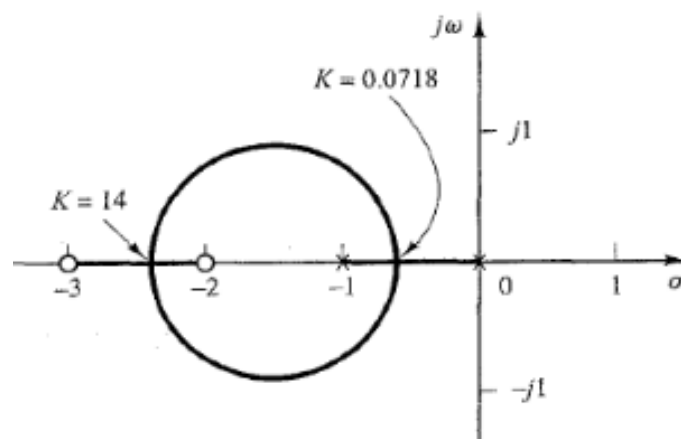
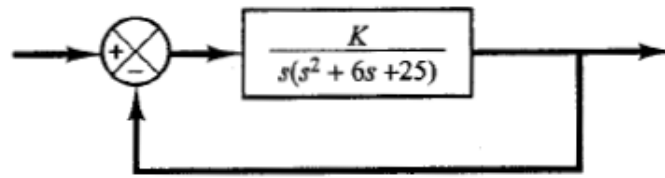


Fig. 16, Root locus of the example

### Example (14)

Draw the root locus of the control system shown below.



$$G(s)H(s) = \frac{K}{S(S^2 + 6S + 25)}$$

- $K = 0$  points: at  $s = 0$ , and  $s = -3 + j4$ .  $s = -3 - j4$

$$\left. \begin{array}{l} s = 0 \\ s = -3 + j4 \\ s = -3 - j4 \end{array} \right\} n = 3$$

- $K = \infty$  points, at

$$\left. \begin{array}{l} s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 0$$

- Number of asymptotes =  $|n - m| = 3 - 0 = 3$
- Angle of Asymptotes  $\theta_0 = 60$ ,  $\theta_1 = 180$ , and  $\theta_2 = 300$
- Intersection of asymptotes with real axis:

$$\sigma = \frac{0 - 3 - 3}{3} = -2$$

- Break away points:

$$G_1(s)H_1(s) = \frac{1}{S(S^2 + 6S + 25)}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{-(3S^2 + 12S + 25)}{S^2(S^2 + 6S + 25)^2} = 0$$

$$3S^2 + 12S + 25 = 0$$

The roots of the above equation are  $s_1 = -2 + j2.0817$  and  $s_2 = -2 - j2.0817$

Notice that at points  $s_1 = -2 + j2.0817$  and  $s_2 = -2 - j2.0817$  the angle condition is not satisfied. Therefore, both points are not used as break away points. Also, if we calculate the value of K at these points, it will be complex value.

- Angle of Departure

$$\theta_D = 180 - 126.87 - 90 = -36.87$$

Using Routh, we can obtain the intersection of root locus with imaginary axis. From Routh array,  $K = 150$  at  $S = \pm j5.0$

The root locus of this example is shown in Fig. 17.

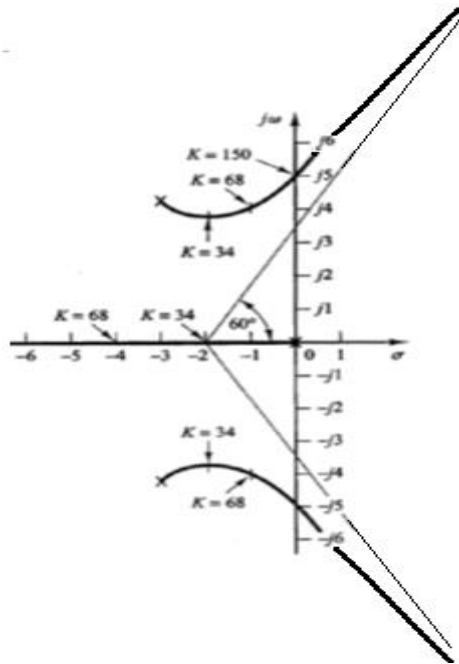
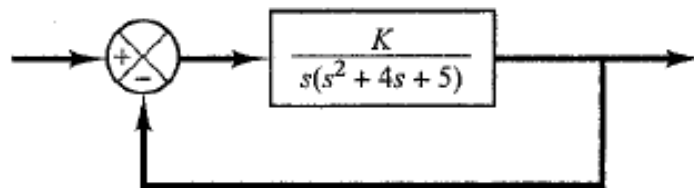


Fig. 17, Root locus of the example.

### Example (15)

Draw the root locus for the control system shown below.



$$G(s)H(s) = \frac{K}{s(s^2 + 4s + 5)}$$

- $K = 0$  points: at  $s = 0$ , and  $s = -2 + j1$ .  $s = -2 - j1$

$$\left. \begin{array}{l} s = 0 \\ s = -2 + j1 \\ s = -2 - j1 \end{array} \right\} n = 3$$

- $K = \infty$  points, at

$$\left. \begin{array}{l} s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 0$$

- Number of asymptotes =  $|n - m| = 3 - 0 = 3$
- Angle of Asymptotes  $\theta_0 = 60^\circ$ ,  $\theta_1 = 180^\circ$ , and  $\theta_2 = 300^\circ$
- Intersection of asymptotes with real axis:



$$\sigma = \frac{0 - 2 - 2}{3} = -1.3333$$

- Break away points:

$$G_1(s)H_1(s) = \frac{1}{S(S^2 + 4S + 5)}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{-(3S^2 + 8S + 5)}{S^2(S^2 + 4S + 5)^2} = 0$$

$$3S^2 + 8S + 5 = 0$$

The roots of the above equation are  $s_1 = -1$  and  $s_2 = -1.6667$

Notice that at points  $s_1 = -1$  the value of K is 2 (+ve real) and at  $s_2 = -1.6667$  the value of K = 1.852 (+ve real). Therefore, both points are used as break away points.

- Angle of Departure

$$\theta_D = 180 - 153.43 - 90 = -63.43$$

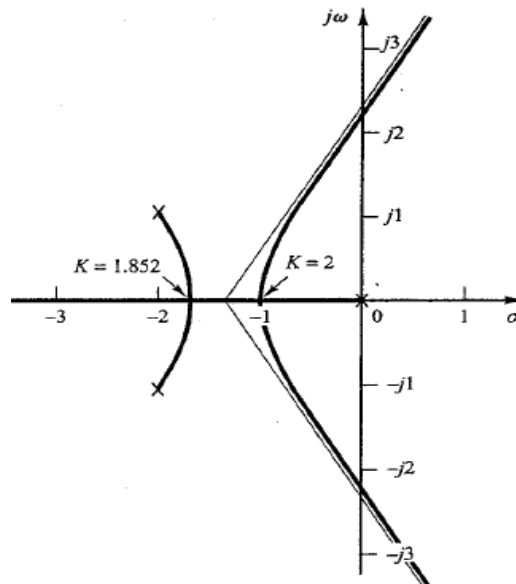


Fig. 18, Root locus of the example.

**Example (16)**

Draw the root locus for the control system whose T.F. is given below,

$$G(s)H(s) = \frac{K(S^2 + 2S + 2)}{S(S + 1)(S + 2)(S^2 + 6S + 13)}$$

- Find the max value of K to make a sustained oscillation system and find the frequency of this oscillation.



b) Calculate the value of K at the breakaway point.

- $K = 0$  points: at  $s = 0$ ,  $s = -1$ , and  $s = -2$  and  $S = -3 + j2$  and  $S = -3 - j2$ .

$$\left. \begin{array}{l} s = 0 \\ s = -1 \\ s = -2 \\ s = -3 - j2 \\ s = -3 + j2 \end{array} \right\} n = 5$$

- $K = \infty$  points, at  $s = -3$ .

$$\left. \begin{array}{l} s = -1 - j1 \\ s = -1 + j1 \\ s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 2$$

- Number of asymptotes =  $|n - m| = 5 - 2 = 3$
- Angle of asymptotes  $\theta_0 = 60$ ,  $\theta_1 = 180$ , and  $\theta_2 = 300$
- Intersection of asymptotes with real axis:

$$\sigma = \frac{(-1 - 2 - 3 - 3) - (-1 - 1)}{3} = -2.3333$$

- Break away point:

$$G_1(s)H_1(s) = \frac{(S^2 + 2S + 2)}{S^5 + 9S^4 + 33S^3 + 51S^2 + 26S}$$

$$\begin{aligned} & \frac{dG_1(s)H_1(s)}{ds} \\ &= \frac{(S^5 + 9S^4 + 33S^3 + 51S^2 + 26S)(2S + 2) - (S^2 + 2S + 2)(5S^4 + 36S^3 + 99S^2 + 102S + 26)}{\{S^5 + 9S^4 + 33S^3 + 51S^2 + 26S\}^2} = 0 \\ & -3S^6 - 26S^5 - 97S^4 - 204S^3 - 274S^2 - 204S - 52 = 0 \\ & S^6 + \frac{26}{3}S^5 + \frac{97}{3}S^4 + 68S^3 + \frac{274}{3}S^2 + 68S + \frac{52}{3} = 0 \end{aligned}$$

Using the function (SOLVE) in your calculator to get the expected breakaway point between 0 & -1, and we found it at  $S = -0.4624$

- Angle of Departure ( $\theta_D$ ) for complex poles

$$\begin{aligned} & \left(180 - \tan^{-1} \frac{1}{2} + 180 - \tan^{-1} \frac{3}{2}\right) - \left(180 - \tan^{-1} \frac{2}{3} + 180 - \tan^{-1} \frac{2}{2} + 180 - \tan^{-1} \frac{2}{1} + 90 + \theta_D\right) = 180 \\ & \theta_D = -30.75 \end{aligned}$$

- Angle of Arrival ( $\theta_A$ ) for complex zeros





$$(90 + \theta_A) - \left(180 - \tan^{-1} \frac{1}{1} + 90 + \tan^{-1} \frac{1}{1} + \tan^{-1} \frac{3}{2} - \tan^{-1} \frac{1}{2}\right) = 180$$

$$\theta_A = 29.75$$

- Intersection of R.L with Imaginary axis:

System characteristic equation is:

$$S^5 + 9S^4 + 33S^3 + 51S^2 + 26S + K(S^2 + 2S + 2) = 0$$

$$S^5 + 9S^4 + 33S^3 + (51 + K)S^2 + (26 + 2K)S + 2K = 0$$

Using Routh-Hurwitz array:

$S^5$	1	33	$26+2K$
$S^4$	9	$51+K$	$2K$
$S^3$	$246-K$	$234+16K$	
$S^2$	$\frac{-K^2 + 51K + 10440}{246 - K}$	$2K$	
$S^1$	$\frac{-18K^3 + 1566K^2 + 57942K + 2442960}{-K^2 + 51K + 10440}$		
$S^0$	$2K$		

From the above array

$$\text{From } S^0 \text{ row} \quad 2K > 0 \quad \rightarrow \quad K > 0 \quad (1)$$

$$\text{From } S^2 \text{ row} \quad 246 - K > 0 \quad \rightarrow \quad K < 246 \quad (2)$$

$$\text{From } S^1 \text{ row} \quad -(K^2 - 51K - 10440) > 0 \quad \rightarrow \quad K^2 - 51K - 10440 < 0$$

$$(K - 130.8103)(K + 79.8103) < 0$$

In that case we have 2 scenarios:

First Scenario

$$(K - 130.8103) > 0 \rightarrow K > 130.8103$$

$$(K + 79.8103) < 0 \rightarrow K < -79.8103$$

$$\text{From } S^1 \text{ row} \quad -18K^3 + 1566K^2 + 57942K + 2442960 > 0$$

$$K^3 - 87K^2 - 3219K - 135720 < 0$$

$$(K - 122.369)(K + 17.685 \pm j28.22) < 0$$

$$(K - 122.369)((K + 17.685)^2 + (28.22)^2) < 0$$

From the above eqn.,

$$K < 122.369 \quad (5)$$

Second Scenario

$$(K - 130.8103) < 0 \rightarrow K < 130.8103 \quad (3)$$

$$(K + 79.8103) > 0 \rightarrow K > -79.8103 \quad (4)$$



From (1) and (4)  $\rightarrow K > 0$

From (2), (3) and (5)  $\rightarrow K < 122.369$

The range of K for stability is

$$0 < K < 122.369$$

The maximum value of K for stability is at  $K = 122.369$

This value is obtained from the row  $S^1$  so the auxiliary equation  $A(S)$  can be obtained from  $S^2$

$$A(s) = \frac{-K^2 + 51K + 10440}{246 - K} S^2 + 2K = 0$$

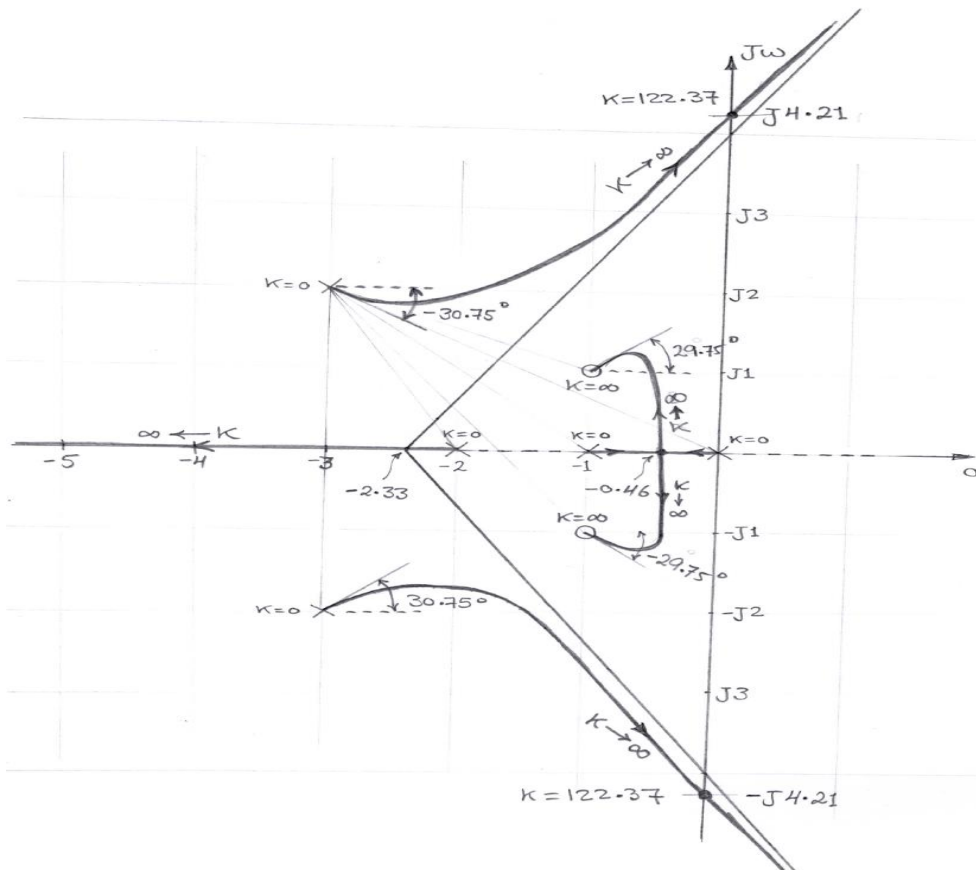
$$A(s) = 13.8 S^2 + 244.738 = 0$$

$$S = \pm j 4.21 \text{ rad/s}$$

To get the value of K at break away point ( $S = -0.4624$ ), substitute with this value of S in the following characteristic eqn.

$$S^5 + 9S^4 + 33S^3 + 51S^2 + 26S + K(S^2 + 2S + 2) = 0$$

$$K = -\frac{S^5 + 9S^4 + 33S^3 + 51S^2 + 26S}{S^2 + 2S + 2} = 3.0956$$







**Example (17)**

Consider the characteristic equation  $S^2 (S + a) + K(S + b) = 0$

Draw root locus of that system when  $b = 1$  and  $a = 10, 9,$  and  $8$

$$G(s)H(s) = \frac{K(S + 1)}{S^2(S + a)}$$

At  $a=10$ ,

- $K = 0$  points: at  $s = 0, s = 0,$  and  $s = -10$

$$\left. \begin{array}{l} s = 0 \\ s = 0 \\ s = -10 \end{array} \right\} n = 3$$

- $K = \infty$  points, at  $s = -1.$

$$\left. \begin{array}{l} s = -1 \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 1$$

- Number of asymptotes =  $|n - m| = 3 - 1 = 2$
- Angle of asymptotes  $\theta_0 = 90,$  and  $\theta_1 = 270$
- Intersection of asymptotes with real axis:

$$\sigma = \frac{(-10) - (-1)}{2} = -4.5$$

- Break away point:

$$G_1(s)H_1(s) = \frac{S + 1}{S^3 + 10S^2}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{(S^3 + 10S^2) - (S + 1)(3S^2 + 20S)}{\{S^3 + 10S^2\}^2} = 0$$

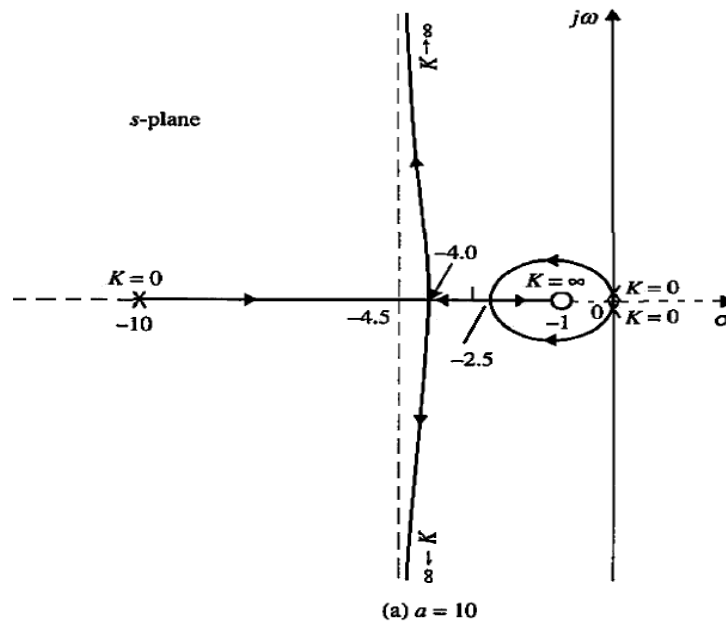
$$-2S^3 - 13S^2 - 20S = 0$$

$$S(S^2 + 6.5S + 10) = 0$$

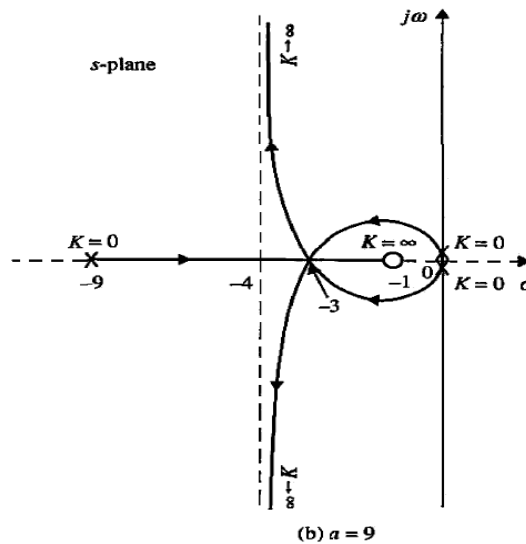
The roots of this eqn. are:

$$\begin{cases} S = 0 & \text{accepted} \\ S = -2.5 & \text{accepted} \\ S = -4.0 & \text{accepted} \end{cases}$$

The root locus is shown in Fig. 19(a)



By the same way: At  $a = 9$ , there are ONE breakaway point at  $S = -3.0$  as shown in Fig. 21(b)



At  $a = 8$ , there is NO breakaway points as shown in Fig. 19(c)

- $K = 0$  points: at  $s = 0$ ,  $s = 0$ , and  $s = -8$

$$\left. \begin{array}{l} s = 0 \\ s = 0 \\ s = -8 \end{array} \right\} n = 3$$

- $K = \infty$  points, at  $s = -1$ .

$$\left. \begin{array}{l} s = -1 \\ s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 1$$

- Number of asymptotes =  $|n - m| = 3 - 1 = 2$



- Angle of asymptotes  $\theta_0 = 90$ , and  $\theta_1 = 270$
- Intersection of asymptotes with real axis:

$$\sigma = \frac{(-8) - (-1)}{2} = -3.5$$

- Break away point:

$$G_1(s)H_1(s) = \frac{S + 1}{S^3 + 8S^2}$$

$$\frac{dG_1(s)H_1(s)}{ds} = \frac{(S^3 + 8S^2) - (S + 1)(3S^2 + 16S)}{\{S^3 + 8S^2\}^2} = 0$$

$$-2S^3 - 11S^2 - 16S = 0$$

$$S(S^2 + 5.5S + 8) = 0$$

The roots of this eqn. are:

$$\begin{cases} S = 0 & \text{accepted} \\ S = -2.75 + j0.66 & \text{rejected} \\ S = -2.75 - j0.66 & \text{rejected} \end{cases}$$

This means, the root locus is as shown below:

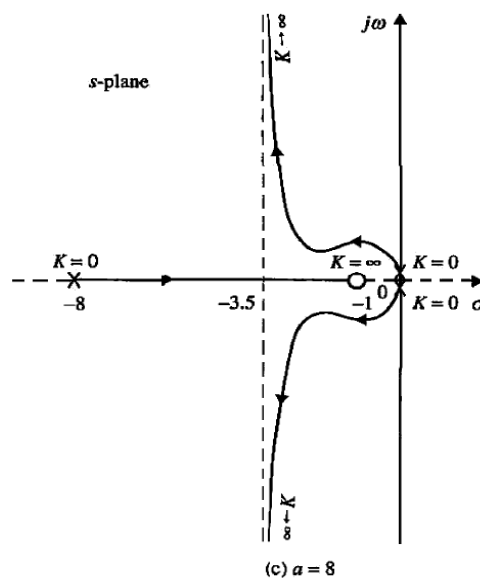


Fig. 19, Root locus of the example.

### Example (18)

Consider the control system shown in Fig. 20. Plot the root locus for this system by



applying the general rules and procedure for constructing root loci, and write the MATLAB code to get root-locus plots.

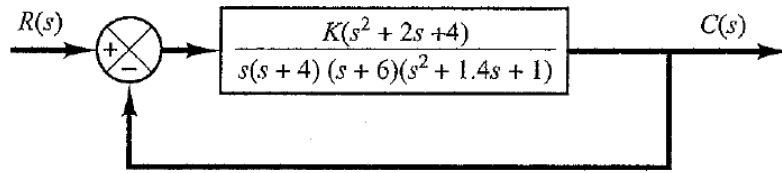


Fig. 20, Control system of example 18.

MATLAB Program shown in Fig. 21 will plot the root-locus diagram for the system. The plot is shown in Fig. 22.

```

1 %This code is used to draw the root locus of TF
2 num = [1 2 4];
3 den = conv([1 0],[1 4]);
4 den = conv(den,[1 6]);
5 den = conv(den,[1 1.4 1]);
6 sys=tf(num,den);
7 rlocus(sys)
    
```

Fig. 21, Matlab code for example 18.

The stability of this system is discussed in chapter (8, stability, Example 21) and we found two ranges of K for stability;

$$0 < K < 15.6106 \quad \& \quad 67.5126 < K < 163.5568$$

At  $K = 15.6106$  &  $K=67.5126$  &  $K=163.5568$  the system oscillates continuously.

These values are obtained from the row  $S^1$  so the auxiliary equation  $A(S)$  is obtained from  $S^2$

$$A(s) = (-K^2 + 143.08K + 14364.56)S^2 + 4K(401 - K) = 0$$

At  $K= 15.6106$ ;

$$16354.43382 S^2 + 24064.63907 = 0 \quad \rightarrow S = \pm j1.213 \text{ rad/s}$$

At  $K= 67.5126$ ;

$$19466.31165 S^2 + 90058.40576 = 0 \quad \rightarrow S = \pm j2.1509 \text{ rad/s}$$

At  $K= 163.5568$ ;

$$11015.44012 S^2 + 155341.8 = 0 \quad \rightarrow S = \pm j3.7553 \text{ rad/s}$$

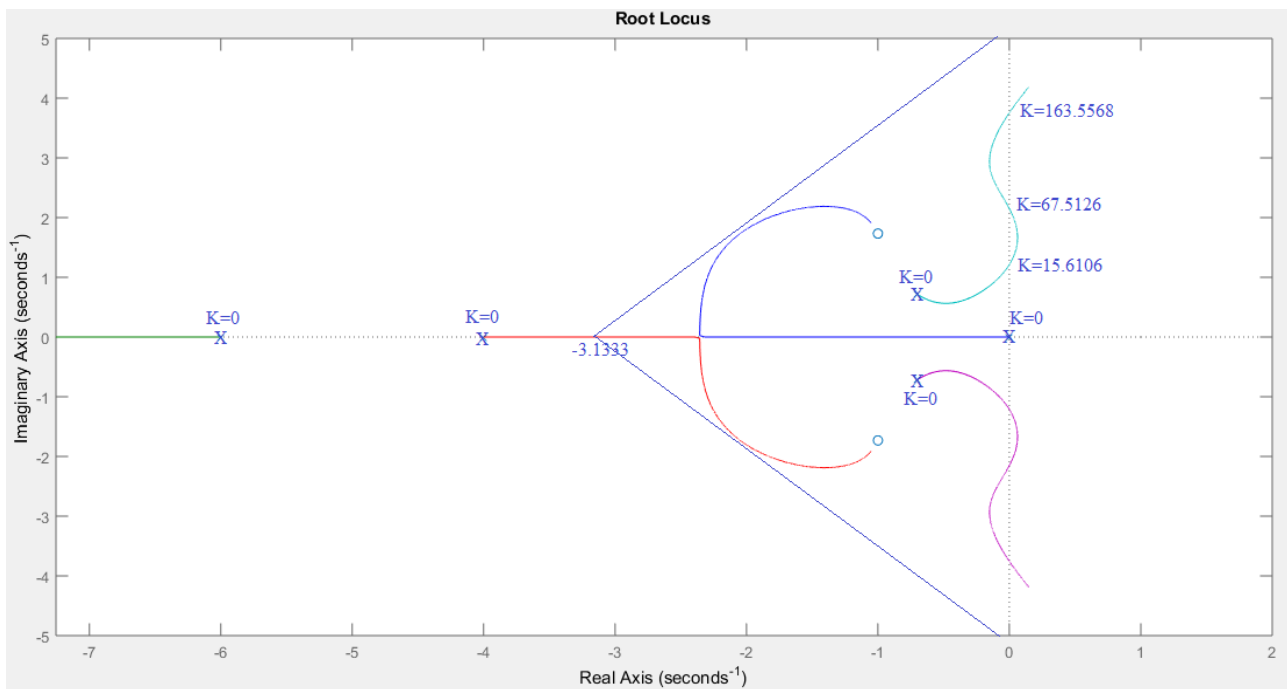


Fig. 22, Matlab root locus obtained from m-code.

**Example 19**

A simplified form of the open-loop transfer function of an airplane with an autopilot in the longitudinal mode is

$$G(s)H(s) = \frac{K(s + a)}{s(s - b)(s^2 + 2\zeta\omega_n s + \omega_n^2)}, \quad a > 0, \quad b > 0$$

Sketch the root locus when  $a = b = 1$ , damping ratio = 0.5, and  $\omega_n = 4$ . Find the range of gain  $K$  for stability.

The open loop TF is

$$G(s)H(s) = \frac{K(s + 1)}{s(s - 1)(s^2 + 4s + 16)}$$

There are 3 Asymptotes at angles of 60 & 180 & -60

$$\sigma = -2/3 = -0.667$$

Breakaway points at  $S = 0.45$  and at  $S = -2.26$

But the points  $S = -0.76 \pm J 2.16$  do not represent break away points

To get intersection of RL with imaginary axis, the characteristic equation is

$$S^4 + 3S^3 + 12S^2 + (K - 16)S + K = 0$$





The stability of this system is discussed in Example (17) in stability chapter and we find the range of K for stability is

$$23.315 < K < 35.685$$

The crossing points on the imaginary axis can be found by solving the auxiliary equation obtained from the  $s^2$  row, that is, by solving the following equation for  $s$ :

$$\frac{52 - K}{3} s^2 + K = 0$$

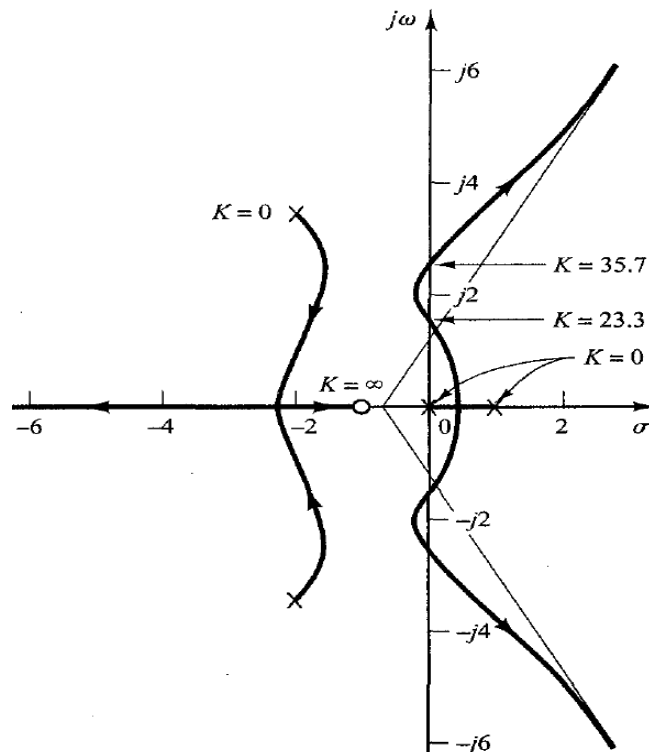
The results are

$$S = \pm j2.56 \quad \text{at } K = 35.685$$

$$S = \pm j1.56 \quad \text{at } K = 23.315$$

The angle of departure ( $\theta_D$ ) of the root locus for the complex poles is calculated as

$$\theta_D = 180 - 120 - 130.5 - 90 + 106 = -54.5$$



### Example 20

The characteristic equation of a control system is

$$S^4 + 12 S^3 + 64 S^2 + 128 S + K = 0$$

- Sketch the system root locus and write the corresponding MATLAB code.
- Find the value of K to give critical-stable system,



c) Find K that gives a critically-damped system.

From the system characteristic equation, we can obtain the system open-loop transfer function by dividing by the term doesn't include K

$$G(S)H(S) = \frac{K}{S^4 + 12S^3 + 64S^2 + 128S}$$

$$G(S)H(S) = \frac{K}{S(S^3 + 12S^2 + 64S + 128)}$$

$$G(S)H(S) = \frac{K}{S(S + 4)(S + 4 + j4)(S + 4 - j4)}$$

**Step#1**

- $K = 0$  points: at  $s = 0$ ,  $s = -4$ , and  $S = -4 + j4$  and  $S = -4 - j4$ .

$$\left. \begin{matrix} s = 0 \\ s = -4 \\ s = -4 - j4 \\ s = -4 + j4 \end{matrix} \right\} n = 4$$

**Step#2**

- $K = \infty$  points: at  $s \rightarrow \infty$ ,  $s \rightarrow \infty$ ,  $s \rightarrow \infty$  and  $s \rightarrow \infty$ .

$$\left. \begin{matrix} s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \\ s \rightarrow \infty \end{matrix} \right\} m = 0$$

**Step#3**

Number of asymptotes =  $|n - m| = 4 - 0 = 4$

**Step#4**

Angle of asymptotes

$$\theta_0 = 45, \theta_1 = 135, \theta_2 = 225 \text{ and } \theta_3 = 315$$

**Step#5**

Intersection of asymptotes with real axis:

$$\sigma = \frac{(0 - 4 - 4 - 4) - 0}{4 - 0} = -3$$

**Step#6**

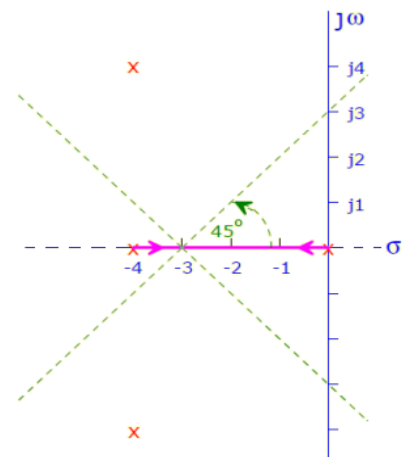
Root locus on real axis is as shown in figure below.

**Step#7**

Break away points:

$$G1(S)H1(S) = \frac{1}{S^4 + 12S^3 + 64S^2 + 128S}$$

$$\frac{dG1(S)H1(S)}{dS} = \frac{-(4S^3 + 36S^2 + 128S + 128)}{(S^4 + 12S^3 + 64S^2 + 128S)^2} = 0$$





$$4S^3 + 36S^2 + 128S + 128 = 0$$

Solving the above equation for break away points, we get:

$$S_1 = -1.5767 \quad (\text{Accepted})$$

$$S_2 = -3.712 + j2.553 \quad (\text{Rejected})$$

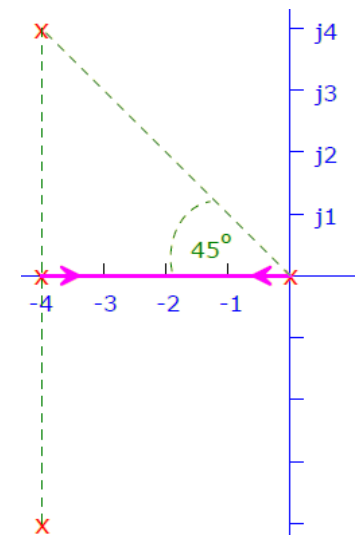
$$S_3 = -3.712 - j2.553 \quad (\text{Rejected})$$

### Step#8

Angle of departure for complex poles:

$$-(\theta_d + 90^\circ + 90^\circ + 135^\circ) = -180^\circ$$

$$\theta_d = -135^\circ$$



### Step#9

Intersection of root locus with imaginary axis:

From Routh based on the system characteristic equation:

		$S^4 + 12S^3 + 64S^2 + 128S + K = 0$	
$S^4$	1	64	K
$S^3$	12	128	
$S^2$	53.33	K	
$S$	$\frac{6826.667 - 12K}{53.33}$		
$S^0$	K		

From the first column of the above Routh array;

$$K > 0$$

$$6826.667 - 12K > 0$$

$$K < 568.889$$

$$0 < K < 568.889$$

Max. value of K for stability  $K = 568.889$  (Critically stable system)

Auxiliary equation

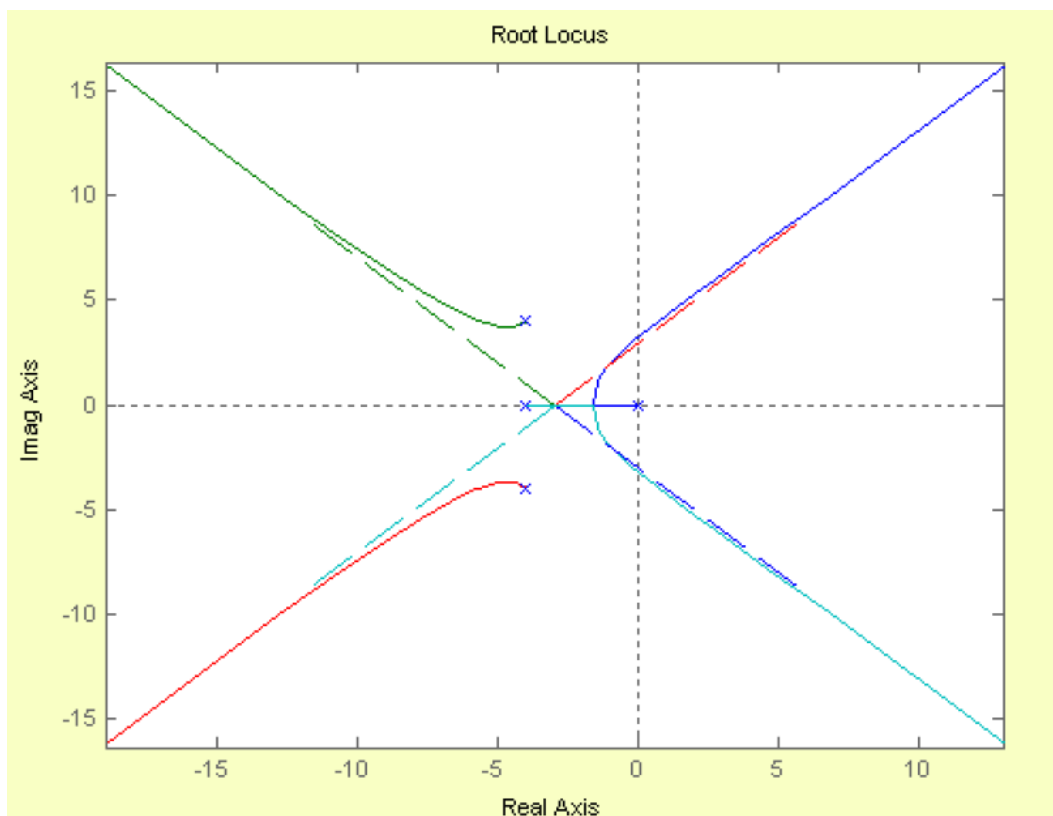
$$53.33 S^2 + 568.889 = 0 \rightarrow S = \pm j 3.266 \text{ (frequency of sustained oscillation)}$$

Matlab Code to draw the root locus of this problem is:



```

Editor - E:\KU Courses\EE3511 Automatic Control\RootLocus1.m
File Edit Text Go Cell Tools Debug Desktop Window Help
- 1.0 + + 1.1 x
1 %This code is used to draw the root locus of TF
2 - Num = [1];
3 - DEN = [1 12 64 128 0];
4 - SYS = tf(Num,DEN)
5 - rlocus(SYS)
script Ln 7 Col 12 OVR
    
```



The value of K for critically damped system is at break away point

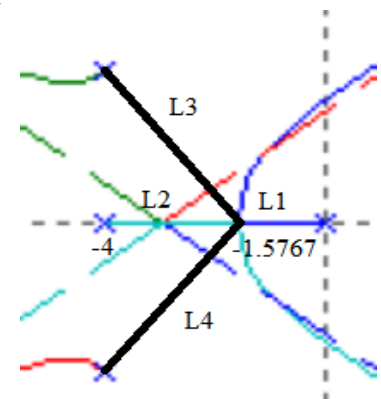
$$K = \frac{\text{حاصل ضرب أطوال الأقطاب}}{\text{حاصل ضرب أطوال الأصفار}} = \frac{L1 \times L2 \times L3 \times L4}{1}$$

$$L1 = 1.5767$$

$$L2 = 4 - 1.5767 = 2.4233$$

$$L3 = L4 = \sqrt{4^2 + 2.4233^2} = 4.6768$$

$$K = \frac{1.5767 \times 2.4233 \times 4.6768 \times 4.6768}{1} = 83.57$$

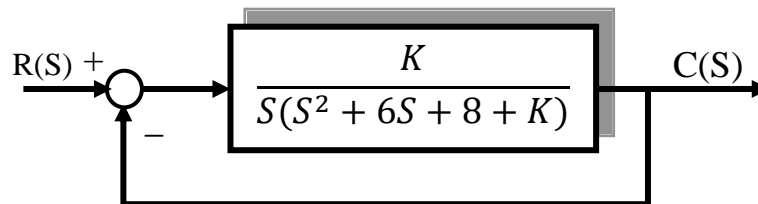
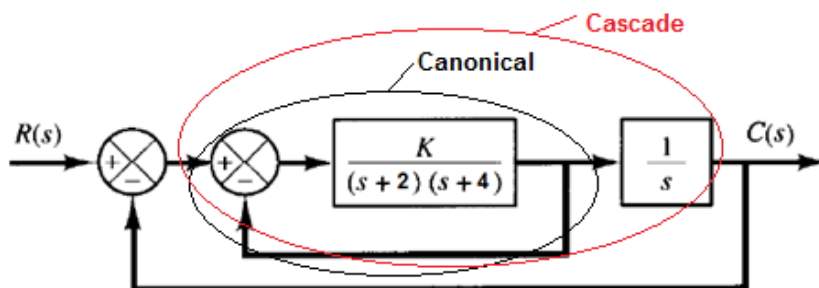
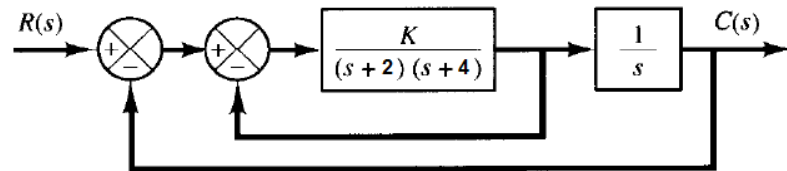




**Example 21:**

For the control system shown in below,

- Sketch the root locus & write the Matlab code,
- Find the maximum value of K to make all poles real,
- Find the value of K to give a damping ratio of 0.707.



Characteristic equation is given as

$$S^3 + 6S^2 + 8S + KS + K = 0$$

$$S^3 + 6S^2 + 8S + K(S + 1) = 0$$

$$GH(s) = \frac{K(S + 1)}{S(S + 2)(S + 4)}$$

- $K = 0$  points: at  $s = 0$ ,  $s = -2$ , and  $s = -4$

$$\left. \begin{array}{l} s = 0 \\ s = -2 \\ s = -4 \end{array} \right\} n = 3$$

- $K = \infty$  points, at  $s = -1$ .

$$\left. \begin{array}{l} s = -1 \\ s = s \rightarrow \infty \\ s \rightarrow \infty \end{array} \right\} m = 1$$

- Number of asymptotes =  $|n - m| = 3 - 1 = 2$
- Angle of asymptotes



$$\theta_0 = 90^\circ \text{ and } \theta_1 = 270^\circ$$

- Intersection of asymptotes with real axis:

$$\sigma = \frac{(-2 - 4) - (-1)}{2} = -2.5$$

- Root Locus on Real axis: As shown in graphics
- Intersection of R.L. with imaginary axis:

First obtain the characteristic equation

$$S^3 + 6S^2 + S(K + 8) + K = 0$$

Using Routh,

$S^3$	1	$K+8$
$S^2$	6	K
$S^1$	$\frac{5K + 48}{6}$	
$S^0$	K	

$$K > 0$$

$$5K + 48 > 0, \text{ this means } K > -9.6$$

This means from both conditions,  $K > 0$

The system is stable for all values of K and there is no intersection of R.L with Imaginary axis

- Breakaway points:

$$\frac{dGH_1}{ds} = \frac{(S^3 + 6S^2 + 8S) - (S + 1)(3S^2 + 12S + 8)}{(S^3 + 6S^2 + 8S)^2}$$

$$S^3 + 4.5S^2 + 6S + 4 = 0$$

$$S_1 = -2.91 \quad (\text{accepted as a breakaway point})$$

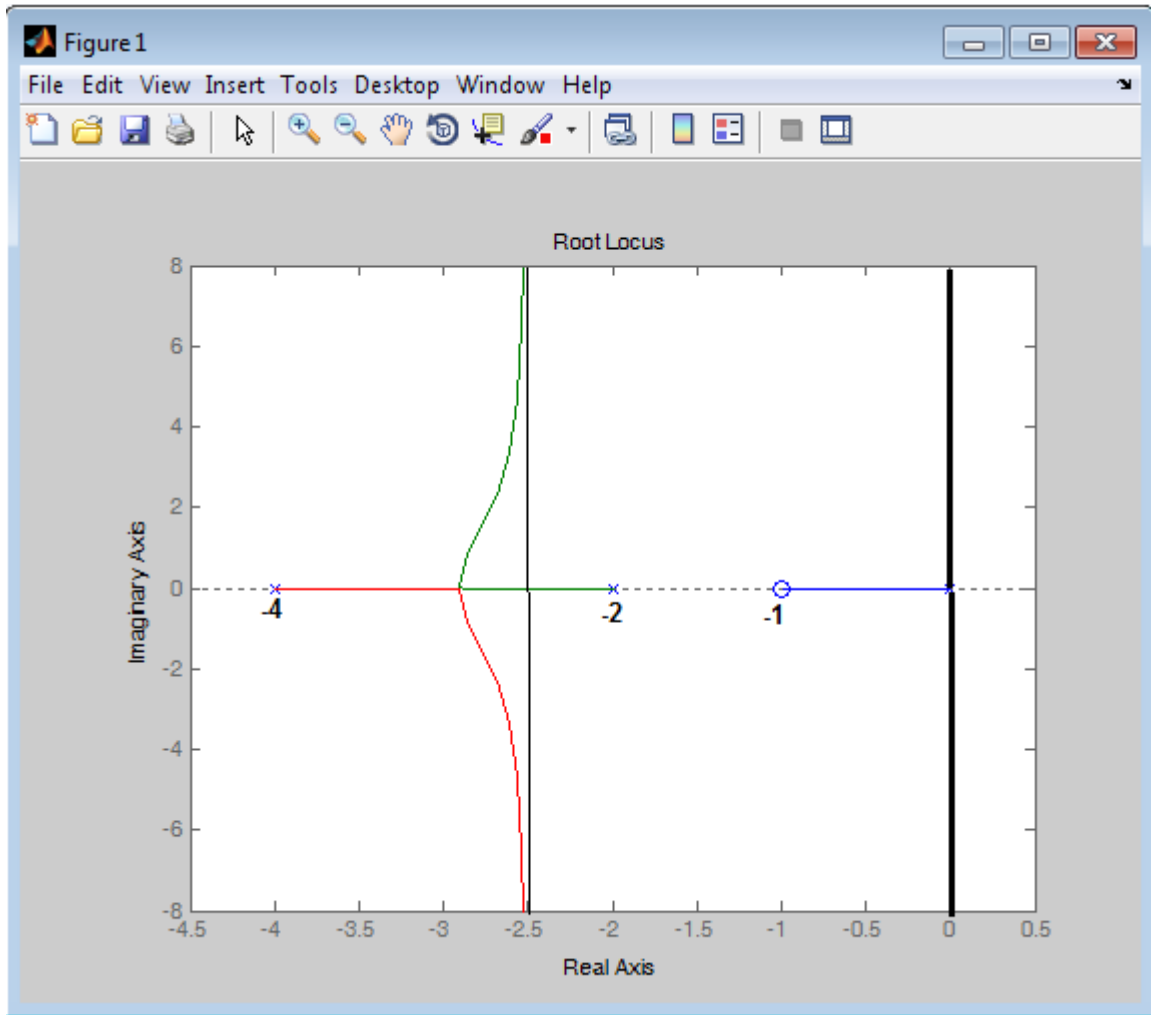
$$S_2 = 0.79 \pm j0.86 \quad (\text{rejected})$$

Root locus of this system is shown below.

The Matlab code is:

```

Editor - E:\SAU Courses\EE3511 Automatic Control\RootLocus4.m
File Edit Text Go Cell Tools Debug Desktop Window Help
1 %This code is used to draw the root locus Final Exam
2 - NUM = [1 1];
3 - DEN = [1 6 8 0];
4 - SYS = tf(NUM,DEN)
5 - rlocus(SYS)
script Ln 3 Col 13 OVR
    
```

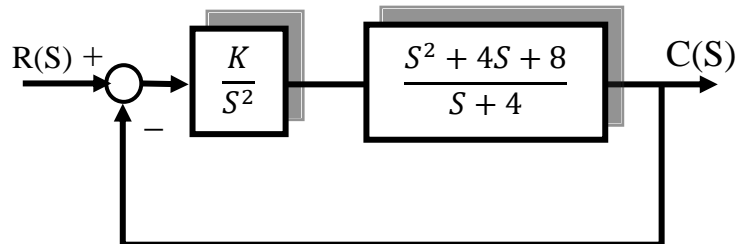


The maximum value of K to make all poles real is at breakaway point

$$K = \frac{2.91 \times 0.91 \times 1.09}{1.91} = 1.511$$

**Example 22:**

For the control system shown below, draw the root locus and find the value of K to give a damping ratio of 0.4, write the Matlab code.



- K = 0 points: at s = 0, s = 0, and s = -4



$$\left. \begin{array}{l} s = 0 \\ s = 0 \\ s = -4 \end{array} \right\} n = 3$$

- $K = \infty$  points, at  $s = -2+j2$  and  $s = -2-j2$ .

$$\left. \begin{array}{l} s = -2 + j2 \\ s = -2 - j2 \\ s \rightarrow \infty \end{array} \right\} m = 2$$

- Number of asymptotes =  $|n - m| = 3 - 2 = 1$
- Angle of asymptotes  
 $\theta_0 = 180$
- Intersection of asymptotes with real axis:  
No Need, since the asymptote is coincident on real axis

- Angle of arrival of complex zeros:  
 $(90 + \theta_A) - (135 + 135 + 45) = 180$   
 $\theta_A = 45$

- Intersection of R.L. with imaginary axis:  
First obtain the characteristic equation

$$S^3 + (K + 4)S^2 + 4KS + 8K = 0$$

Using Routh,

$s^3$	1	4K
$s^2$	K+4	8K
$s^1$	$\frac{4K^2+8K}{K+4}$	
$s^0$	8K	

Then  $K > 0$  also  $K > -4$  and  $K > -2$

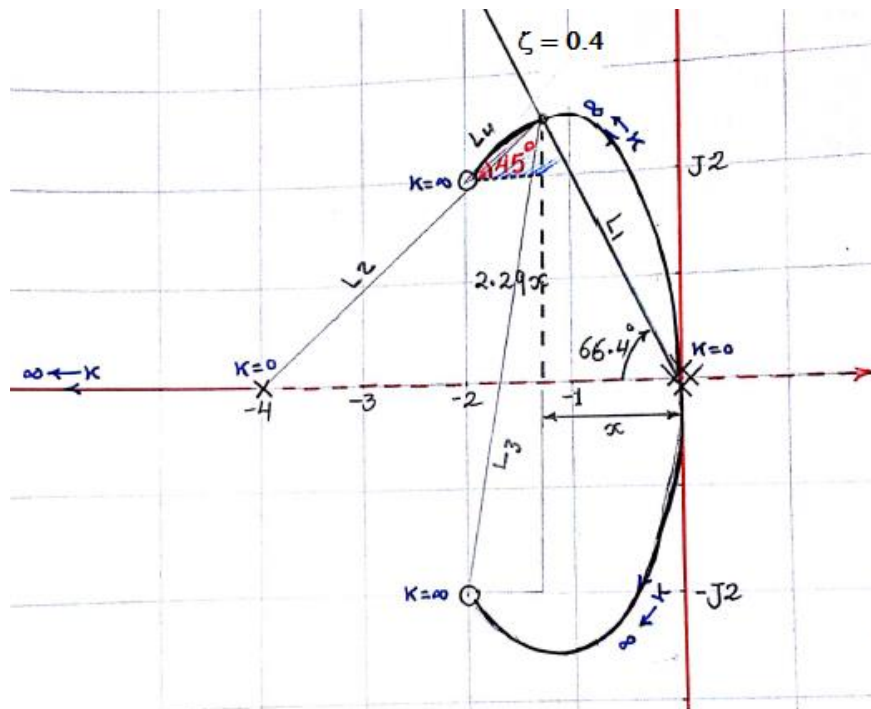
Then we find that  $K > 0$

This means that the system is stable for all values of  $K$

This indicate that there no intersection with imaginary axis

Root locus of this system is shown below.





Since  $\xi = 0.4$ , then  $\beta = \cos^{-1}(0.4) = 66.4^\circ$ . we draw a line at angle  $\beta$  intersect with RL as shown above. Applying the angle condition,

$$\tan^{-1} \frac{2.29x + 2}{2 - x} + \tan^{-1} \frac{2.29x - 2}{2 - x} - \left\{ 113.6 + 113.6 + \tan^{-1} \frac{2.29x}{4 - x} \right\} = 180$$

$$\tan^{-1} \frac{2.29x + 2}{2 - x} + \tan^{-1} \frac{2.29x - 2}{2 - x} = 47.2 + \tan^{-1} \frac{2.29x}{4 - x}$$

Taking *tan* for both sides:

$$\frac{\frac{2.29x + 2}{2 - x} + \frac{2.29x - 2}{2 - x}}{1 - \frac{2.29x + 2}{2 - x} \times \frac{2.29x - 2}{2 - x}} = \frac{1.08 + \frac{2.29x}{4 - x}}{1 - \frac{2.4732x}{4 - x}}$$

$$\frac{(2 - x)(2 + 2.29x + 2.29x - 2)}{(2 - x)^2 - (5.2441x^2 - 4)} = \frac{1.08(4 - x) + 2.29x}{4 - x - 2.4732x}$$

$$\frac{9.16x - 4.58x^2}{-4.2441x^2 - 4x + 8} = \frac{1.21x + 4.32}{4 - 3.4732x}$$

$$21.0427 x^3 - 26.96 x^2 + 44.24x - 34.56 = 0$$

Solving this equation for  $x$ , we find that

$x_1 = 0.9261$  (accepted)

$x_2 = 0.1776 + j1.3198$  (rejected)

$x_3 = 0.1776 - j1.3198$  (rejected)

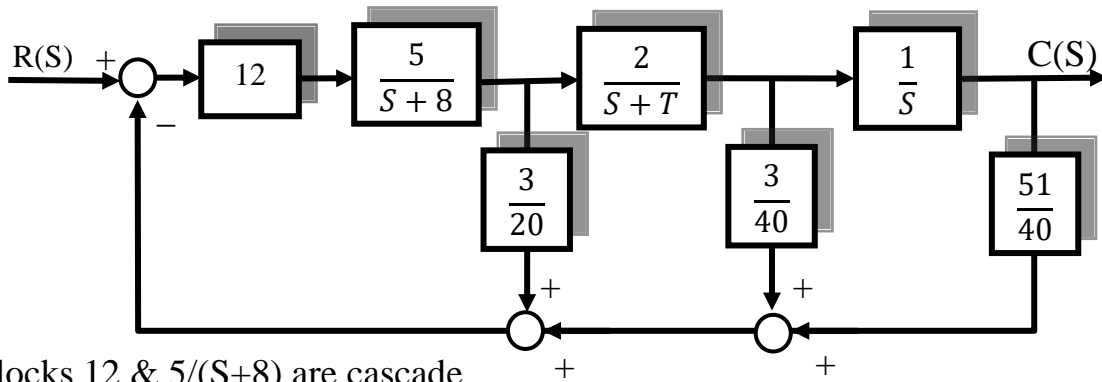




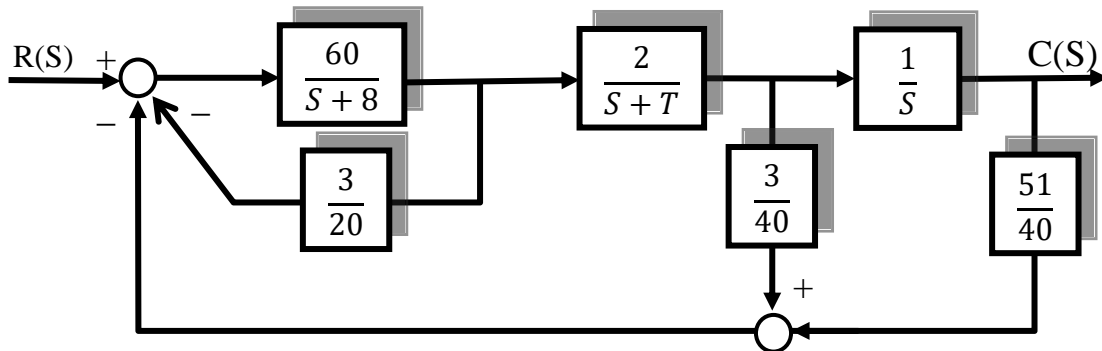
**Example 23:**

For the closed-loop control system shown below,

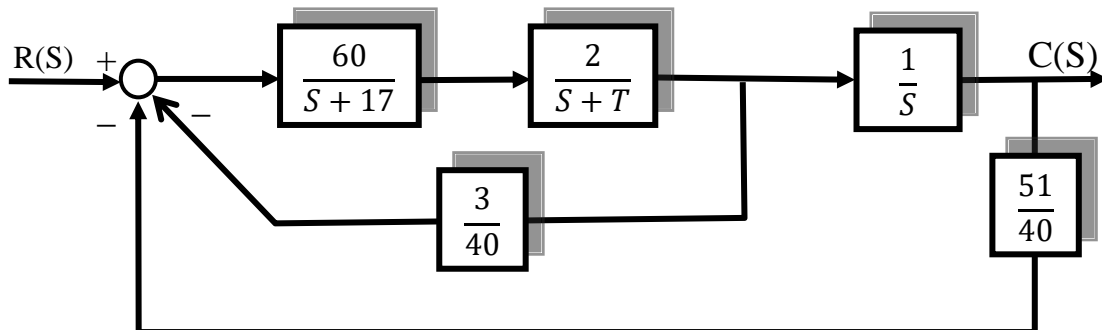
- Using block diagram algebra, find the system transfer function  $C(S)/R(S)$ .
- Obtain the system characteristic equation.
- Draw the root locus as the system gain  $T$  change from  $0$  to  $\infty$ , and write the Matlab<sup>®</sup> code,
- Find the minimum value of  $T$  such that all poles are real,
- Find the value of  $T$  that gives a damping ratio of  $0.707$



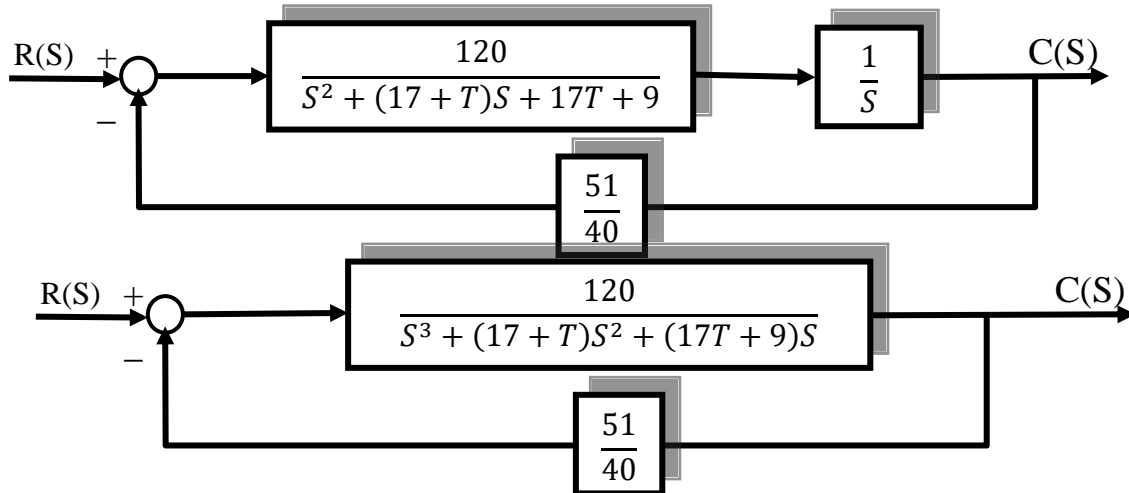
The blocks  $12$  &  $5/(s+8)$  are cascade



The blocks  $60/(s+8)$  &  $3/20$  are canonical



The blocks  $60/(s+17)$  cascaded with  $2/(s+T)$  and the result is canonical with  $3/40$



$$\frac{C(S)}{R(S)} = \frac{120}{S^3 + (17 + T)S^2 + (17T + 9)S + 153}$$

The system characteristic equation is

$$S^3 + (17 + T)S^2 + (17T + 9)S + 153 = 0$$

Rearrange the above equation to be:

$$S^3 + 17S^2 + TS^2 + 17TS + 9S + 153 = 0$$

$$S^3 + 17S^2 + 9S + 153 + TS(S + 17) = 0$$

To obtain the open loop T.F. divide by the term that doesn't include T

$$G(S)H(S) = \frac{TS(S + 17)}{S^3 + 17S^2 + 9S + 153} = \frac{TS(S + 17)}{(S + 17)(S + j3)(S - j3)}$$

$$G(S)H(S) = \frac{TS}{(S + j3)(S - j3)}$$

Steps to draw Root Locus:

**Step#1**

- $T = 0$  points: at  $s = j3, s = -j3$ .

$$\left. \begin{matrix} s = -j3 \\ s = j3 \end{matrix} \right\} n = 2$$

**Step#2**

- $T = \infty$  points: at  $S=0, s \rightarrow \infty$ ,

$$\left. \begin{matrix} s = 0 \\ s \rightarrow \infty \end{matrix} \right\} m = 1$$

**Step#3**

Number of asymptotes =  $|n - m| = 2 - 1 = 1$



**Step#4**

Angle of asymptotes

$$\theta_0 = 180$$

**Step#5**

Intersection of asymptotes with real axis: (No need)

**Step#6**

Root locus on real axis is as shown in figure below.

**Step#7**

Break away points:

$$G1(S)H1(S) = \frac{S}{(S^2 + 9)}$$

$$9 - S^2 = 0 \rightarrow S^2 = 9 \rightarrow S_1 = -3 \text{ (accepted) and } S_2 = 3 \text{ (rejected)}$$

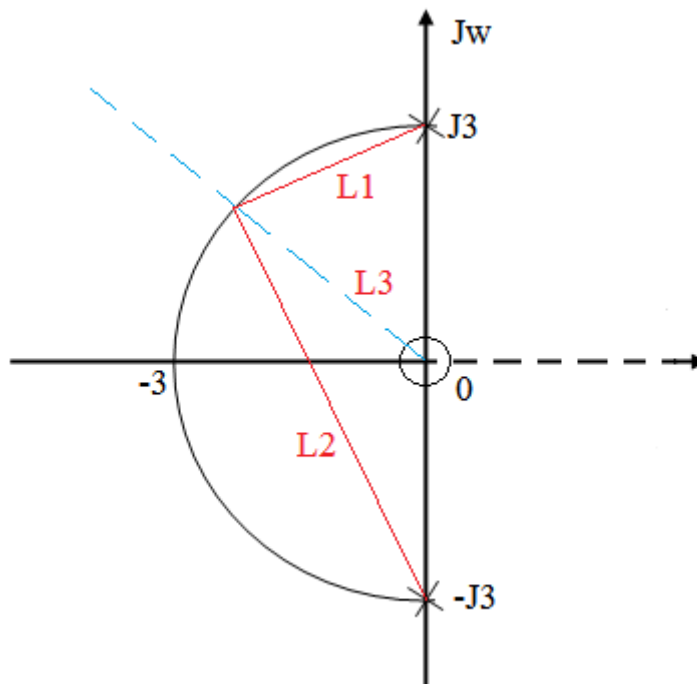
**Step#8**

Angle of departure ( $\theta_d$ ) :

$$90 - (90 + \theta_d) = 180 \rightarrow \theta_d = -180$$

**Step#9**

Intersection of root locus with imaginary axis (No need)



The minimum value of T for all poles are real is at break away point.

$$L1 = L2 = \sqrt{3^2 + 3^2} = 4.2426$$

$$L3 = 3$$



$$T = \frac{4.2426 \times 4.2426}{3} = 6$$

the value of T that gives a damping ratio of 0.707 ( $\theta=45$ )

Angle condition at the point of intersection of the line of  $45^\circ$  with root locus

$$135 - \left( 180 - \tan^{-1} \frac{3-x}{x} + 90 + \tan^{-1} \frac{x}{3+x} \right) = 180$$

$$\tan^{-1} \frac{3-x}{x} + \tan^{-1} \frac{x}{3+x} = 45$$

$$\frac{\frac{3-x}{x} + \frac{x}{3+x}}{1 - \frac{3-x}{x} \times \frac{x}{3+x}} = 1$$

$$\frac{(3-x)(3+x) + x^2}{x(3+x) - x(3-x)} = \frac{9}{2x^2} = 1$$

$$x = \sqrt{4.5} = 2.1213$$

$$L1 = \sqrt{2.1213^2 + (3 - 2.1213)^2} = 2.2961$$

$$L2 = \sqrt{2.1213^2 + (3 + 2.1213)^2} = 5.5433$$

$$L2 = 2.1213\sqrt{2} = 3$$

$$T = \frac{2.2961 \times 5.5433}{3} = 4.2427$$

**Example 24:**

A control system with unity feedback has a forward open loop transfer function:

$$G(S) = \frac{K(S^2 - 2S + 2)}{S(S + 2)}$$

- Draw the root locus for this system
- Write the Matlab code.
- Find the value of K to make all poles equal
- Find the value of K to give a damping ratio of 0.707 and find the location of roots

- $K = 0$  points: at  $s = 0, s = -2$

$$\left. \begin{array}{l} S = 0 \\ S = -2 \end{array} \right\} n = 2$$



- $K = \infty$  points, at  $s = 1+j1$  and  $s=1 - j1$ .

$$\left. \begin{matrix} 1 - j1 \\ 1 + j1 \end{matrix} \right\} m = 2$$

- Number of asymptotes =  $|n - m| = 2 - 2 = 0$  (no asymptotes)
- Angle of asymptotes                      None
- Intersection of asymptotes with real axis:                      None
- Root Locus on real axis : As shown in next page
- Angle of Arrival for complex zeros  $\theta_A$  :  
from the angle condition,  
 $(90 + \theta_A) - (45 + \tan^{-1} 1/3) = 180 \rightarrow \theta_A = 153.4^\circ$
- Intersection of Root Locus with Imag. Axis:

Characteristic equation is  
 $S^2 (K+1) + S(2-2K) + 2K = 0$   
Using Routh Array:

$S^2$	K+1	2K
$S^1$	2-2K	0
$S^0$	2K	

from which we can obtain that

$$2K > 0 \quad K > 0$$

$$2-2K > 0 \quad K < 1$$

$$K+1 > 0 \quad K > -1$$

the range of stability is  $0 < K < 1$

**at  $K = 1$  the system is critically stable**

$$\text{auxiliary equation } A(S) = 2S^2 + 2 = 0 \rightarrow S = \pm j1$$

- Break away points

$$\frac{dGH1}{dS} = \frac{(S^2 + 2S)(2S - 2) - (S^2 - 2S + 2)(2S + 2)}{den^2}$$

$$S^2 - S - 1 = 0$$

$$S = 1.6 \text{ (Rejected)}$$

and

$$S = -0.61 \text{ (Accepted)}$$

b) The roots are equal at break away point ( $S = -0.61$ )

From the magnitude condition:

$$|K| = \frac{0.61 \times 1.39}{\sqrt{1 + 1.61^2}} = 0.236$$

c) To get damping ratio of 0.707, draw a line with angle  $45^\circ$  intersect with the root locus at point A, that is located at  $x$  distance on the X-axis



from the angle condition,

$$135 + 180 + \tan^{-1} \frac{1-x}{1+x} - 135 - \tan^{-1} \frac{x}{2-x} = 180$$

$$\tan^{-1} \frac{1-x}{1+x} = \tan^{-1} \frac{x}{2-x}$$

taking tan for both sides:

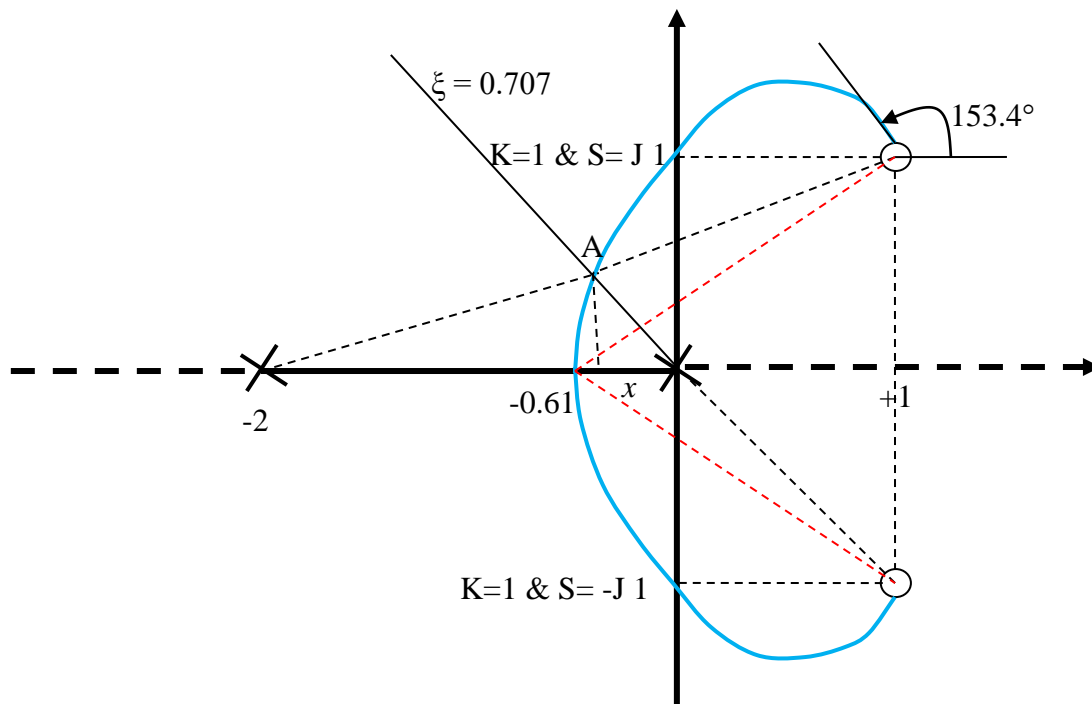
$$\frac{1-x}{1+x} = \frac{x}{2-x}$$

$$4x = 2 \rightarrow x = 0.5$$

from the magnitude condition,

$$|K| = \frac{0.5\sqrt{2} \times \sqrt{0.5^2 + 1.5^2}}{\sqrt{0.5^2 + 1.5^2} \sqrt{1.5^2 + 1.5^2}} = 0.333$$

The location of roots is at  $S = 0.5 + J 0.5$  and  $S = 0.5 - J 0.5$

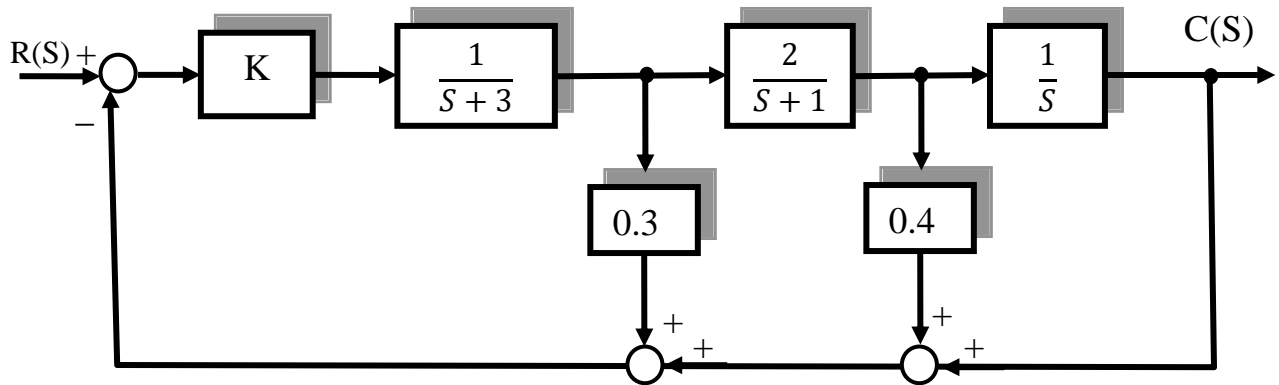


**Example 25:**

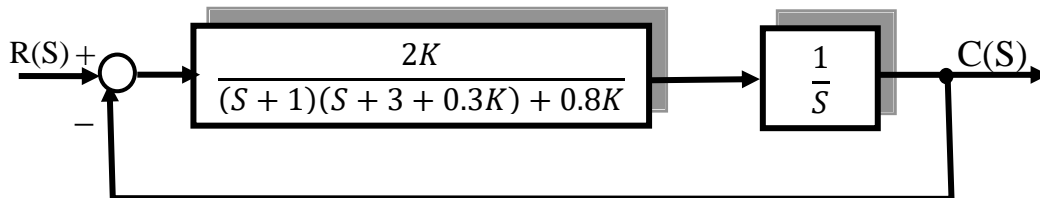
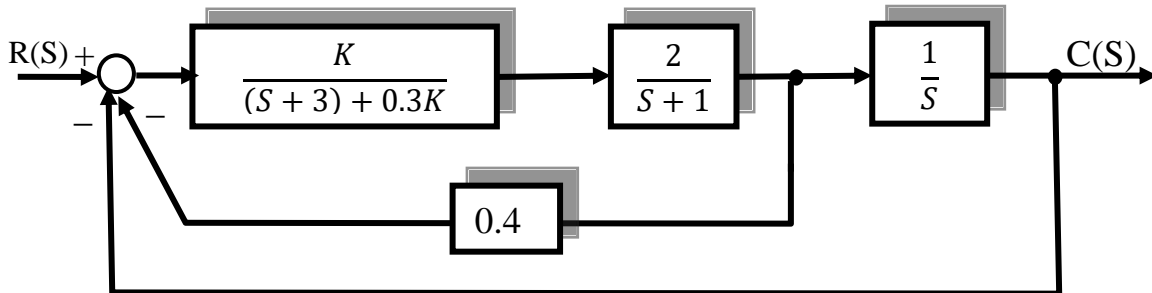
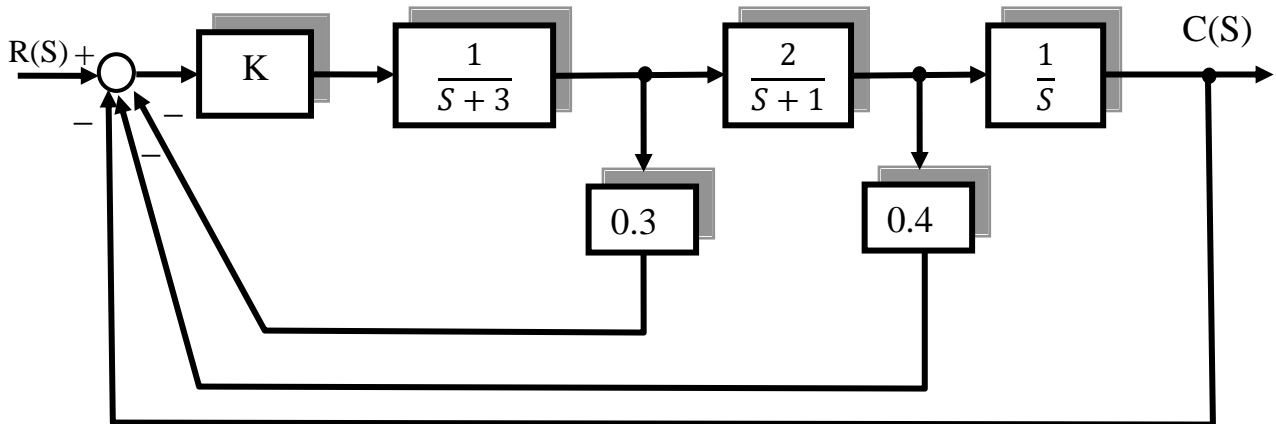
For the control system shown below,

- Calculate the transfer function  $C(s)/R(s)$ ,
- Plot the root locus as the gain  $K$  changes from 0 to  $\infty$ , and **write the Matlab code**
- Determine the value of  $K$  that necessary to place the real root at  $S = -5$  and then determine the damping ratio for the conjugate complex roots.
- Determine the maximum value of  $K$  to give critically damped system, then find the location of roots.





The block diagram can be rearranged as:



Therefore, the closed loop T.F. is:

$$\frac{C(S)}{R(S)} = \frac{2K}{S\{(S+1)(S+3+0.3K) + 0.8K\} + 2K}$$

The system characteristic equation is given as:

$$S\{(S+1)(S+3+0.3K) + 0.8K\} + 2K = 0$$



$$S^3 + 4S^2 + 3S + 0.3K(S^2 + 3.667S + 6.667) = 0$$

$$G(S)H(S) = \frac{0.3K(S + 1.834 + j1.82)(S + 1.834 - j1.82)}{S(S + 1)(S + 3)}$$

Let  $0.3K = k$

- $k = 0$  points: at  $s = 0, s = -1, s = -3$

$$\left. \begin{array}{l} S = 0 \\ S = -1 \\ S = -3 \end{array} \right\} n = 3$$

- $k = \infty$  points, at  $s = -1.834 + j1.82$  and  $s = -1.834 - j1.82$

$$\left. \begin{array}{l} 1.834 + j1.82 \\ 1.834 - j1.82 \end{array} \right\} m = 2$$

- Number of asymptotes =  $|n - m| = 3 - 2 = 1$  (one asymptote)

- Angle of asymptote  $\theta = 180^\circ$

- Intersection of asymptotes with real axis:

Since one asymptote with angle  $180^\circ$ , no meaning for intersection

- Root Locus on real axis : As shown in next page

- Angle of Arrival for complex zeros  $\theta_A$  :

from the angle condition,

$$(90 + \theta_A) - (\tan^{-1} 1.82/1.166 + 180 - \tan^{-1} 1.82/0.834 + 180 - \tan^{-1} 1.82/1.834) = 180 \rightarrow \theta_A = 37.2^\circ$$

- Intersection of Root Locus with Imag. Axis: No Intersection

- Break away points

$$\frac{dGH1}{dS} = \frac{(S^3 + 4S^2 + 3S)(2S + 3.667) - (S^2 + 3.667S + 6.667)(3S^2 + 8S + 3)}{den^2}$$

$$= 0$$

$$2S^4 + 8S^3 + 6S^2 + 3.667S^3 + 14.668S^2 + 11S$$

$$- \{3S^4 + 8S^3 + 3S^2 + 11S^3 + 29.336S^2 + 11S + 20S^2 + 53.336S + 20\} = 0$$

$$S^4 + 7.333S^3 + 31.668S^2 + 53.336S + 20 = 0$$

$$S = -0.515$$

c) To locate a pole at ( $S=-5$ ), connect all poles and zeros to that point

From the magnitude condition:

$$|k| = \frac{2 \times 4 \times 5}{\sqrt{1.82^2 + 3.166^2} \times \sqrt{1.82^2 + 3.166^2}} = 3$$

$$0.3 K = 3 \rightarrow K = 3/0.3 = 10 \quad \text{###}$$



Substitute the value of K in the characteristic equation

Then  $S^3 + 7S^2 + 14S + 20$

But there is a pole at  $S = -5$

Divide the characteristic equation by  $S+5$ , we get  $S^2 + 2S + 4$

That gives the equation of the other two poles

$S_1 = -1+J\sqrt{3}$  and  $S_2 = -1 -J\sqrt{3}$

To get the damping ratio of these conjugate complex poles, calculate the angle  $\beta$

$\beta = \tan^{-1} \sqrt{3}/1 = 60^\circ$

the damping ratio  $\xi = \cos(60) = 0.5$  #####

d) To get critically damped system,  $\xi = 1$ , This is happened at break away point.

From the magnitude condition:

$$|k| = \frac{0.515 \times 0.485 \times 2.485}{\sqrt{1.319^2 + 1.82^2} \times \sqrt{1.319^2 + 1.82^2}} = 0.123$$

$0.3 K = 0.123 \rightarrow K = 0.123/0.3 = 0.41$  ###

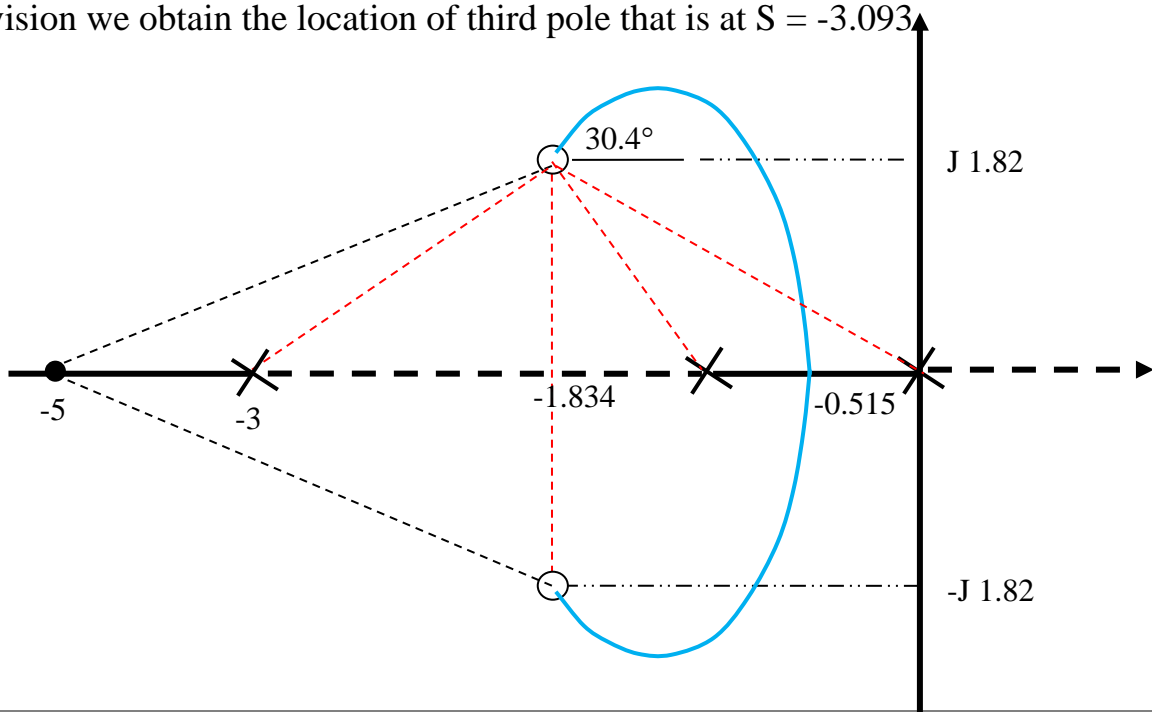
Substitute the value of K in the characteristic equation

Then  $S^3 + 4.123S^2 + 3.451S + 0.82$

at break away point there are two equal poles with equation

$(S+0.515)^2 = S^2 + 1.03 S + 0.265$

by division we obtain the location of third pole that is at  $S = -3.093$





**Example 26:**

The closed-loop T.F. of a control system is

$$\frac{C(S)}{R(S)} = \frac{K(S + 1)}{S^2 + (K - 3)S + K}$$

Sketch the system root locus and write the corresponding MATLAB code then find

- The value of K to give continuous-oscillated system and find the location of poles.
- The value of K to give a critically-damped system and find the location of poles.
- The value of K to give a damping ratio of 0.707 system and find the location of poles.
- The value of K to locate one pole at  $S = -5$  and find the location of other pole.

The closed-loop T.F. of a control system is

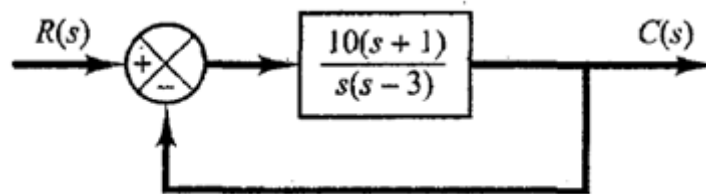
$$\frac{C(S)}{R(S)} = \frac{K(S + 1)}{S^2 + (K - 3)S + K}$$

The system characteristic equation is  $S^2 + (K - 3)S + K = 0$

$$S(S - 3) + K(S + 1) = 0$$

$$GH(S) = \frac{K(S + 1)}{S(S - 3)}$$

And the system block diagram is as shown below



K=0 points

At  $S = 0$  and at  $S = 3$

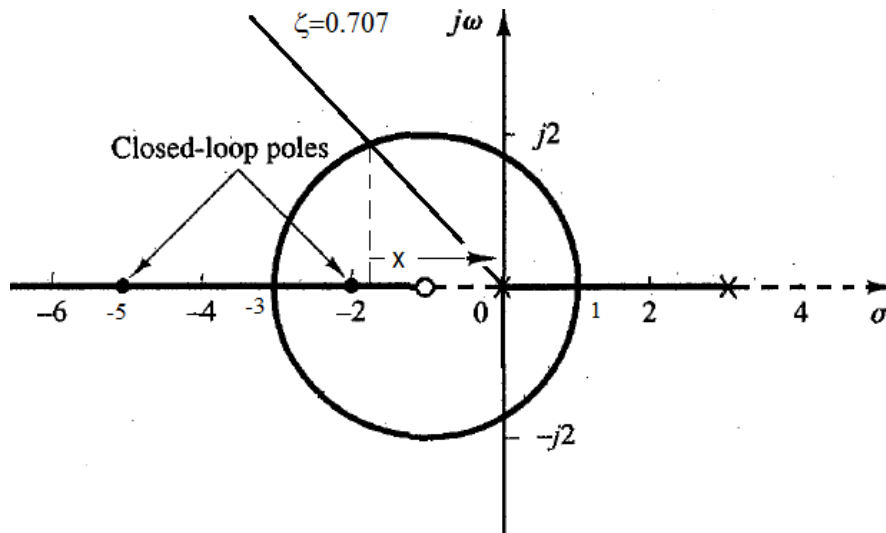
K =  $\infty$  points

At  $S = -1$

There is only ONE asymptotes at angle = 180

No need for the intersection of asymptotes with real axis

Root locus on real axis is as shown in Fig.



Break away points

$$G_1H_1(S) = \frac{S + 1}{S(S - 3)}$$

$$\frac{dG_1H_1(S)}{ds} = \frac{(S^2 - 3S) - (S + 1)(2S - 3)}{\{S(S - 3)\}^2} = 0$$

$$S^2 + 2S - 3 = 0$$

$$(S+3)(S-1) = 0$$

$$S_1 = 1 \text{ (accepted as break away point)}$$

$$S_2 = -3 \text{ (accepted as break in point)}$$

Intersection of root locus with the imaginary axis

$$\text{The system characteristic equation is } S^2 + (K - 3)S + K = 0$$

Using routh

$$S^2 \quad 1 \quad K$$

$$S^1 \quad k-3$$

$$S^0 \quad K$$

$$K > 0 \quad (1)$$

$$K - 3 > 0 \rightarrow K > 3 \quad (2)$$

From (1) and (2)  $K > 3$

$$\text{Auxiliary equation is } S^2 + 3 = 0 \rightarrow S = \pm j\sqrt{3}$$

a) The value of K to give continuous-oscillated system and find the location of poles.



Is obtained from Routh array  $K = 3$ , the location of roots are at  $S_{1,2} = \pm j\sqrt{3}$

b) The value of  $K$  to give a critically-damped system and find the location of poles.

Is obtained at the break in point  $K = 6 \times 3/2 = 9$ . And the location of roots at  $S_{1,2} = -3$

c) The value of  $K$  to give a damping ratio of 0.707 system and find the location of poles.

By drawing a line with 45 as shown in Fig. the intersection of this line with the root locus is located at  $x$

Applying the angle condition

$$180 - \tan^{-1} \frac{x}{x-1} - \left( 135 + 180 - \tan^{-1} \frac{x}{3+x} \right) = 180$$

$$\tan^{-1} \frac{x}{3+x} - \tan^{-1} \frac{x}{x-1} = 315$$

Taking tan for both sides

$$\frac{\frac{x}{3+x} - \frac{x}{x-1}}{1 + \frac{x^2}{(3+x)(x-1)}} = -1$$

$$2x^2 + 2x - 3 = 4x$$

$$2x^2 - 2x - 3 = 0$$

$$x_1 = 1.8229 \text{ (accepted)}$$

$$x_2 = -0.823 \text{ (rejected at it is -ve value)}$$

$$K = \frac{(1.8229\sqrt{2})(\sqrt{1.8229^2 + 4.8229^2})}{\sqrt{1.8229^2 + 0.8229^2}} = 6.646$$

To find the location of poles at this value of  $K$ , the characteristic equation is

$$S^2 + 3.646S + 6.646 = 0$$

$$S_{1,2} = 1.8229 \pm j 1.8229$$

d) The value of  $K$  to locate one pole at  $S = -5$  and find the location of other pole.

To locate pole at  $S = -5$ , the value of  $K = 5 \times 8/4 = 10$

To find the location of other pole at this value of  $K$ , the characteristic equation is

$$S^2 + 7S + 10 = 0$$

$$S_1 = -5 \text{ and } S_2 = -2$$

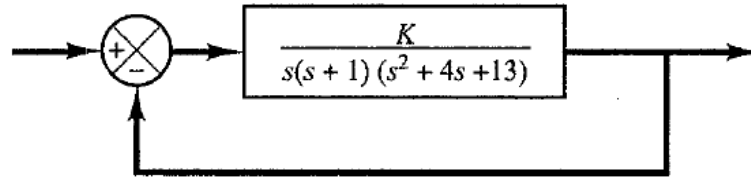
### Example 27:

The characteristic equation of a control system is



$$S^4 + 5S^3 + 17S^2 + 13S + K = 0$$

- Sketch the system root locus and write the corresponding MATLAB code
  - Find The value of K to give critical-stable system and find the location of poles
  - Find K that gives a system with a damping ratio = 1, and find the location of poles.
- First we obtain the loop T.F. GH(S) as shown in figure below.



**Solution.** The open-loop poles are located at  $s = 0, s = -1, s = -2 + j3,$  and  $s = -2 - j3$ . A root locus exists on the real axis between points  $s = 0$  and  $s = -1$ . The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k + 1)}{4} = 45^\circ, -45^\circ, 135^\circ, -135^\circ$$

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 1 + 2 + 2}{4} = -1.25$$

The breakaway and break-in points are found from  $dK/ds = 0$ . Noting that

$$K = -s(s + 1)(s^2 + 4s + 13) = -(s^4 + 5s^3 + 17s^2 + 13s)$$

we have

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 34s + 13) = 0$$

from which we get

$$s = -0.467, \quad s = -1.642 + j2.067, \quad s = -1.642 - j2.067$$

Point  $s = -0.467$  is on a root locus. Therefore, it is an actual breakaway point. The gain values  $K$  corresponding to points  $s = -1.642 \pm j2.067$  are complex quantities. Since the gain values are not real positive, these points are neither breakaway nor break-in points.

The angle of departure from the complex pole in the upper half  $s$  plane is



$$\theta = 180^\circ - 123.69^\circ - 108.44^\circ - 90^\circ$$

or

$$\theta = -142.13^\circ$$

Next we shall find the points where root loci may cross the  $j\omega$  axis. Since the characteristic equation is

$$s^4 + 5s^3 + 17s^2 + 13s + K = 0$$

Using Routh

$S^4$	1	17	K
$S^3$	5	13	
$S^2$	14.4	K	
$S^1$	$\frac{187.2-5K}{14.4}$		
$S^0$	K		

$$K > 0 \quad (1)$$

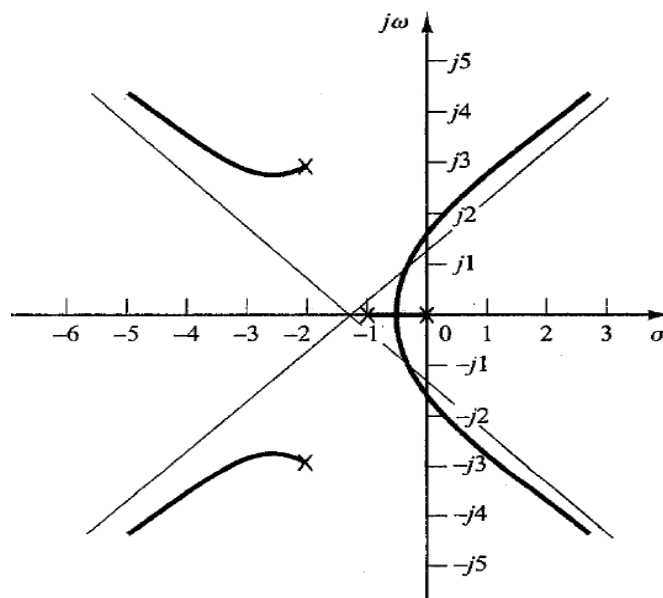
$$187.2 - 5K > 0, \text{ therefore } K < 37.44 \quad (2)$$

$$0 < K < 37.44$$

$$\text{Auxiliary equation } A(S) = 14.4 S^2 + 37.44 = 0$$

$$S = \pm j 1.6125$$

The root locus of the control system is shown below



a)





The value of K to give critical-stable system and find the location of poles.

From Routh array,  $K = 37.44$

There are two poles at  $S = \pm J1.6125$  i.e. its equation is  $S^2 + 2.6$

To know the other two poles, substitute the value of K in the characteristic equation

$$S^4 + 5 S^3 + 17 S^2 + 13 S + 37.44 = 0$$

By long division

$$\frac{S^4 + 5 S^3 + 17 S^2 + 13 S + 37.44}{S^2 + 37.44} = S^2 + 5S + 14.4$$

Then the other two roots at

$$S = -2.5 \pm J 2.85482$$

b)The value of K to give a system with a damping ratio = 1, and find the location of poles.

This is at break away point  $(-0.467)$

Substitute this value of S in the characteristic equation

$$\begin{aligned} (-0.467)^4 + 5 (-0.467)^3 + 17 (-0.467)^2 + 13 (-0.467) + K &= 0 \\ -2.825162 + K &= 0 \end{aligned}$$

$$K = 2.825162$$

There are two poles at  $S = (-0.467)$  there equation is  $S^2 + 0.934S + 0.2181$

To know the other two poles, substitute the value of K in the characteristic equation

$$S^4 + 5 S^3 + 17 S^2 + 13 S + 2.825162 = 0$$

By long division

$$\frac{S^4 + 5 S^3 + 17 S^2 + 13 S + 2.825162}{S^2 + 0.934 S + 0.2181} = S^2 + 4.07S + 12.9$$

Then the other two roots at

$$S = -2.1 \pm J2.99$$



### 3. Complementary Root Locus Method (Positive Feedback Systems)

In large control systems, there may be a positive-feedback inner loop as shown in Fig. 23. This inner loop is usually stabilized by a main outer loop. Therefore, the root locus of this positive feedback is discussed in this section.

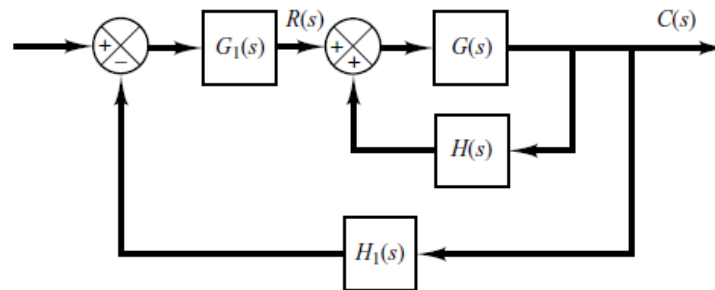


Fig. 23 Positive feedback control system

The closed-loop transfer function of the inner (+ve feedback) loop is

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 - KG(s)H(s)}$$

The characteristic equation of this system is;

$$1 - KG(s)H(s) = 0 \rightarrow KG(s)H(s) = +1$$

We can deal with such positive feedback system as a negative feedback system with negative gain  $K$  ( $-\infty < K < 0$ ). Therefore, we can use the nine steps mentioned in section 2 with some modifications and the obtained locus is called complementary (inverse) root locus. Considering that the properties of inverse root locus such as (Starting & Ending points, Number of branches and Symmetry) are same as root locus.

While the magnitude condition doesn't change, the angle conditions is modified as:

#### Angle condition :

$$\theta = \sum \text{angle of zeros} - \sum \text{angle of poles} = 0$$

$$\theta = \sum_{i=1}^{i=m} \angle(s + Z_i) - \sum_{j=1}^{j=m} \angle(s + P_j) = 0$$



Where a zero angle  $\angle(s + Z_i)$  is the angle measured from +ve real axis of a vector drawn from the zero located at  $Z_i$  to the point  $s$  at which the unknown angle is calculated.

Also, a pole angle  $\angle(s + P_j)$  is the angle measured from +ve real axis of a vector drawn from the pole located at  $P_j$  to the point  $s$  at which the unknown angle is calculated.

Regarding the 9 steps for plotting inverse root locus, here we will write the steps that are altered:

- **Step #4 Angle of Asymptotes**

we can calculate the angles of asymptotes by

$$\theta_k = \frac{k(360)}{|n - m|}$$

Where  $k = 0, 1, 2, \dots, |n - m| - 1$

Substituting  $k=0$  we get the angle of 1<sup>st</sup> asymptote

Substituting  $k=1$  we get the angle of 2<sup>st</sup> asymptote, ... etc.

In case of number of asymptotes =2, therefore the angles are  $\theta_0 = 0$  and  $\theta_1 = 180$

In case of number of asymptotes =3, therefore the angles are  $\theta_0 = 0$ ,  $\theta_1 = 120$  and  $\theta_2 = 240$  or  $-120$

In case of number of asymptotes =4, therefore the angles are  $\theta_0 = 0$ ,  $\theta_1 = 90$ ,  $\theta_2 = 180$  and  $\theta_3 = 270$  or  $-90$

- **Step #7 Angles of Departure for complex poles and/or Angles of Arrival for complex zeros**

The angle of departure or arrival of an inverse root locus at a pole or zero, respectively, can be calculated from the angle condition, such that:

Consider the pole-zero map given in Fig. 24, the angle of departure ( $\theta_D$ ) of the complex pole is calculated as:

$$\tan^{-1} \frac{1}{1} - \left\{ 180 - \tan^{-1} \frac{1}{1} + 90 + \tan^{-1} \frac{1}{2} + \theta_D \right\} = 0$$

$$\theta_D = 153.4349$$

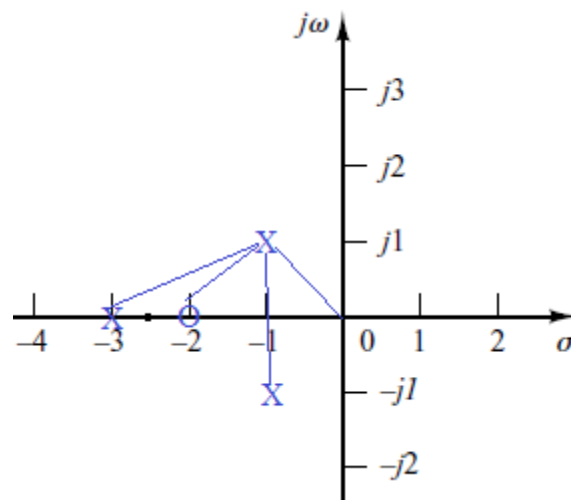


Fig. 24, Calculation of angle of departure for inverse root locus



**Sheet 8 (Root Locus Analysis)**

1) Sketch the root locus diagram for the following control systems:

$$G(s)H(s) = \frac{K(S + 2)}{(S + 1)(S + 4)}$$

$$G(s)H(s) = \frac{K}{S(S + 6)(S + 8)}$$

$$G(s)H(s) = \frac{K}{S(S + 1)(S + 3)(S + 4)}$$

$$G(s)H(s) = \frac{K}{S(S + 2)(S + 10)^2}$$

$$G(s)H(s) = \frac{K(S + 1)(S + 2)}{(S + 0.1)(S - 1)}$$

$$G(s)H(s) = \frac{K(S + 5)}{(S + 2)(S + 4)}$$

$$G(s)H(s) = \frac{K}{(S + 1)(S + 2)(S + 4)}$$

$$G(s)H(s) = \frac{K}{S^2(S + 9)}$$

$$G(s)H(s) = \frac{K(S + 8)}{(S + 14)(S^2 + 20S + 110)}$$

$$G(s)H(s) = \frac{K(S + 3)}{S(S^2 + 2S + 2)(S + 5)(S + 6)}$$

2) Given the following characteristic equations;

$$S^3 + 5S^2 + (6+K)S + 8K = 0$$

$$S^3 + 3S^2 + (K+2)S + 10K = 0$$

Sketch the root locus then find

- The value of K for all roots to be real, and find the location of roots
- The value of K to give a critically damped system, and find the location of roots
- The value of K to make  $\zeta=0.707$ , and find the location of roots
- Find the maximum value of K for stability

3) A control system has a forward and feedback T.F. as

$$G(s) = \frac{K}{S(S^2 + 4S + 13)}, \quad H(s) = 1$$

Sketch the root locus then find

- The value of K to give a damping ratio of 0.2588
- Find the location of the roots at that damping ratio
- The value of K that gives a marginally stable system

4) A unity feedback control system whose forward T.F is



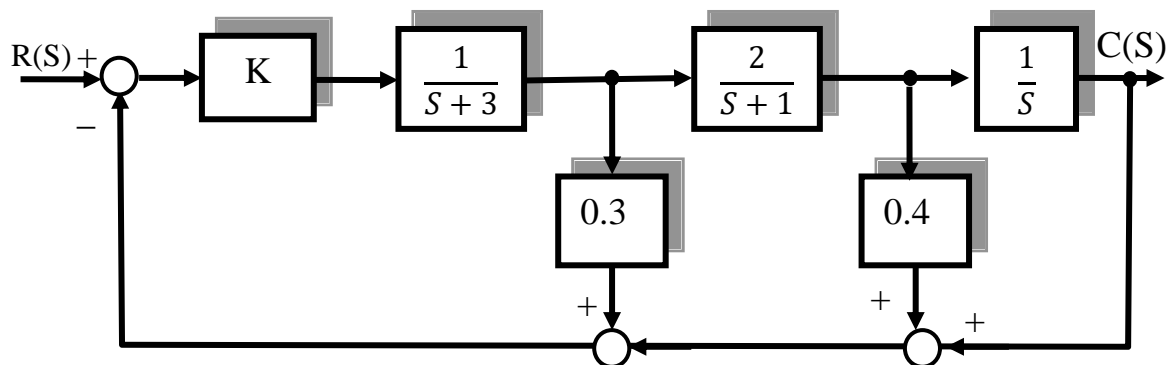
$$G(s) = \frac{K(S^2 + 14S + 74)}{(S + 3)(S + 5)(S + 10)}$$

Sketch the root locus then find

- Maximum value of K that make all roots real, then find the location of roots
- The value of K that make the real root at  $S = -14$ , then find the corresponding damping ratio
- The value of K that gives minimum damping ratio, then find the location of roots

5) For the control system shown below

- Plot the root locus
- Determine the value of K to place the real root at  $S = -5$ , then determine the damping ratio
- Determine the maximum value of K so that  $\zeta = 1$ , then find the location of roots



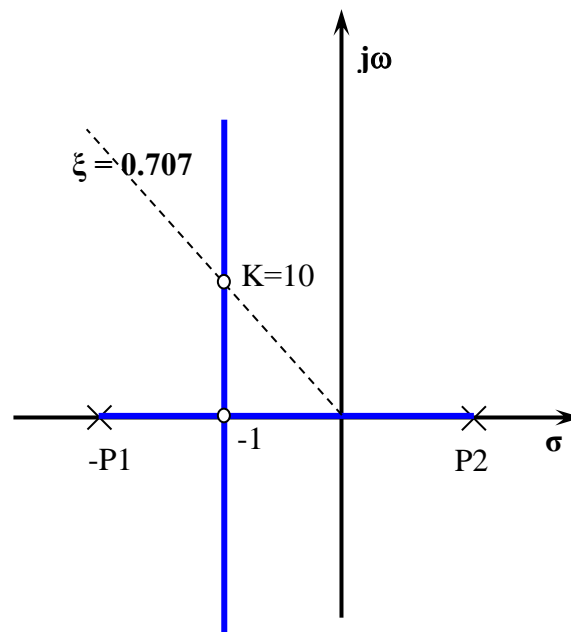
6) A control system has a closed loop T.F. with unity feedback as

$$\frac{C(s)}{R(s)} = \frac{K(S + 3)}{S(S + 2) + K(S + 3)}$$

Draw the R.L. and design the system to have a damping ratio of 0.866, then find the location of roots.

7) The root locus of the two poles P1 and P2 is shown in Fig. 2. It is found that at  $K=10$ , the damping ratio is 0.707

- Find the two roots P1 and P2
- Find the value of K to give  $\zeta = 0.5$ , then find the location of roots



7) Draw the root locus for the control system whose T.F. is given below,

$$G(s)H(s) = \frac{K (S^2 + 2S + 2)}{S (S + 1)(S + 2)(S^2 + 6S + 13)}$$

- c) Find the max value of K to make a sustained oscillation system and find the frequency of this oscillation.
- d) Calculate the value of K at the breakaway point.

**Hint:** There is only ONE breakaway point at  $S = -0.46$

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