CHAPTER # 6 MODELING OF PHYSICAL SYSTEMS

<u>After completing this chapter, the students will be able to:</u>

- Find the transfer function for linear, time-invariant electrical networks,
- Obtain the transfer function for linear, time-invariant translational mechanical systems, and draw its mechanical networks,
- Obtain the transfer function for linear, time-invariant rotational mechanical systems (with gear train and without gear train),
- Obtain the transfer function for linear, time-invariant electromechanical systems.

1. Introduction

This chapter presents mathematical modeling of mechanical systems, electrical systems and electromechanical systems.

Mechanical systems can be either *<u>translational</u>* or *<u>rotational</u>*. Although the fundamental relationships for both types are derived from Newton's law, they are different enough to warrant separate considerations.

Any physical system consists of mechanical elements. There are three types of basic elements in such kind of systems:

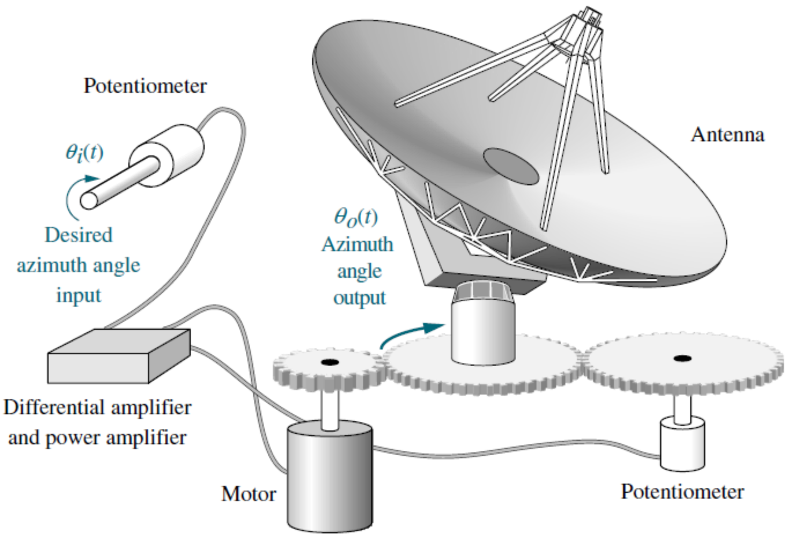
Translational Motion

- Mass elements
- Linear Spring elements
- Linear Dampers elements

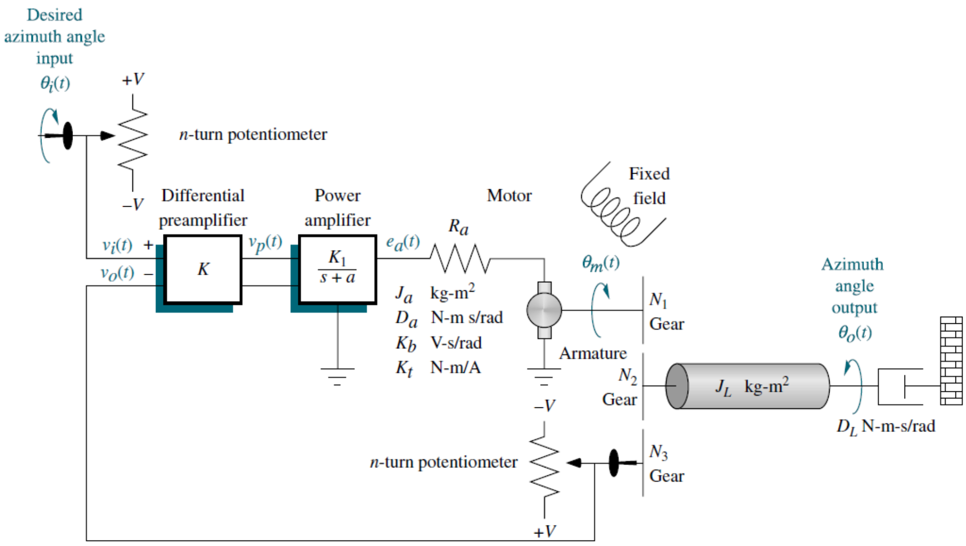
Rotational Motion

- Moment of Inertia elements
- Torsional Spring elements
- Torsional Damper elements

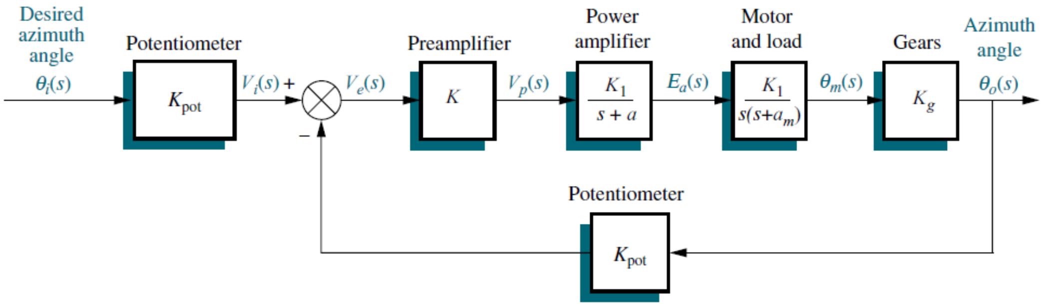
Example of physical system that has rotary motion is the Antenna Azimuth Position Control System shown in Figure below.



(a) Layout of the system



(b) schematic diagram of the system

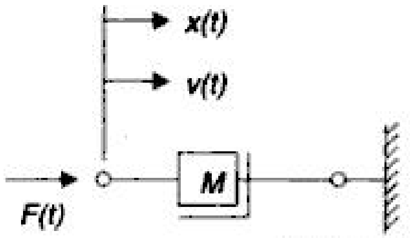


(c) Block diagram of the system

2. Mass / Inertia element

Newton's law (translational motion): If a force (F) is acting on rigid body through the center of mass (M) in a given direction, the acceleration (a) of the rigid body in the same direction is directly proportional to the force acting on it and is inversely proportional to the mass of the body. That is,

$$acceleration\ (a) = \frac{Force\ (F)}{Mass\ (M)} \quad OR \qquad F = M \times a = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$

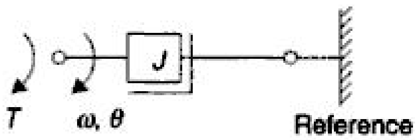


Suppose that there are many forces acting on a body of mass, then

$$\sum F = M \times a$$

Newton's law (Rotational motion):

$$angular\ acceleration\ (\alpha) = \frac{Torque\ (T)}{Inertia\ (J)} \quad OR\ T = J \times \alpha = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$



Suppose that there are many torques acting on a rotating body of inertia, then

$$\sum T = J \times \alpha$$

3. Spring / Torsional Spring element

A linear spring is a mechanical element that can be deformed by external force or torque such that the deformation is directly proportional to the force or torque applied to the element.

For translational motion shown in Fig. 1, the force that arises in the spring is proportional to *x* and is given by:

$$F = k\,x$$

where *x* is the elongation of the spring and *k* is a proportionality constant called the ***spring constant* or** (stiffness) and has units of [force/displacement]=[N/m] in SI units.

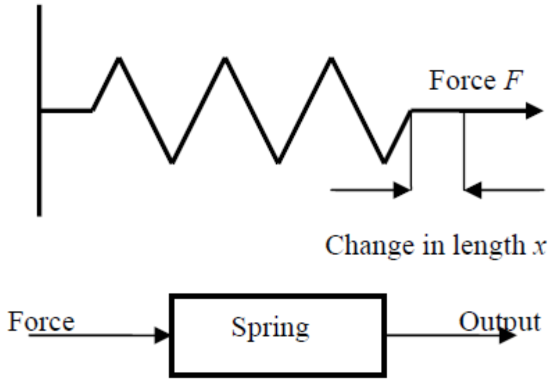
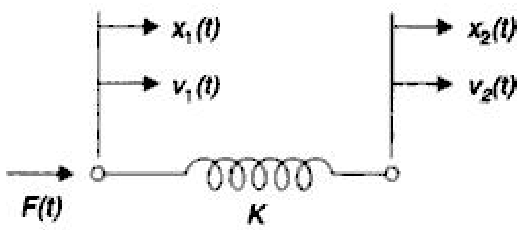


Fig. 1, Linear Spring

If the spring is free to move at its 2nd end, then:

$$F = k\,x_1 - k\,x_2$$



Consider the torsional spring shown in Fig. 2, where one end is fixed and a torque T is applied to the other end. The ***angular displacement*** of the free end is θ. The torque *T* in the torsional spring is:

$$T = k\,\theta$$

where θ is ***the angular displacement*** and k is the ***spring constant or*** (stiffness) for torsional spring and has units of [Torque/angular displacement]=[N-m/rad] in SI units.

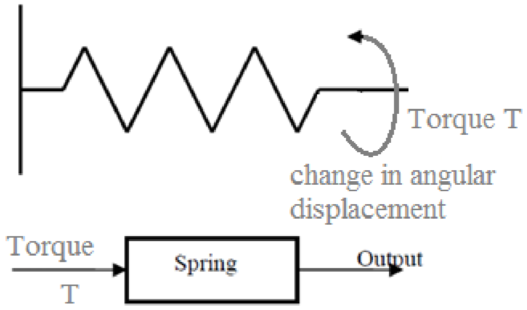
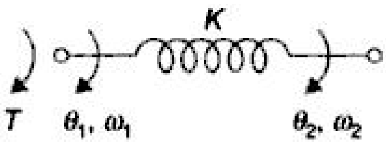


Fig. 2, Torsional Spring

If the spring is free to move at its 2nd end, then:

$$T = k\ \theta_1 - k\ \theta_2$$



4. Damper (Dashpot)

A damper is a mechanical element that dissipates energy in the form of heat instead of storing it. Figure 4 shows a schematic diagram of a translational damper, or a dashpot that consists of a piston and an oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by oil.

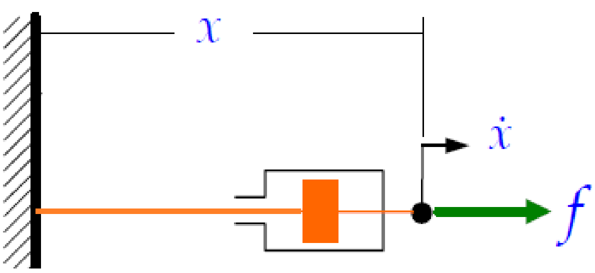


Fig. 4, Translational Damper

In the damper, the damping force F that arises in it is proportional to the velocity,

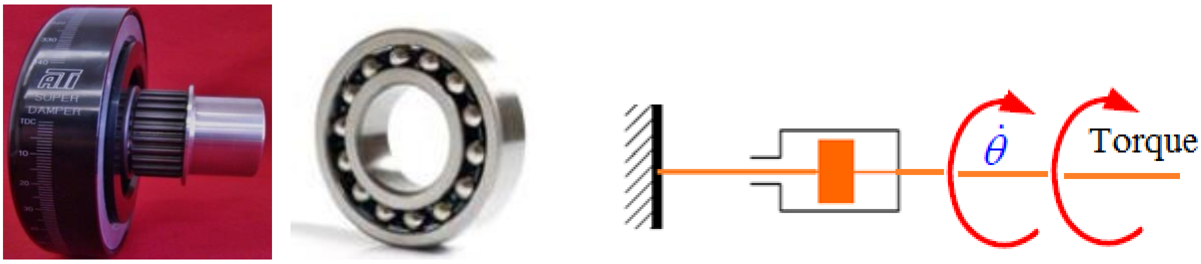
$$F = B\ \dot{x}$$

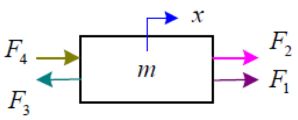
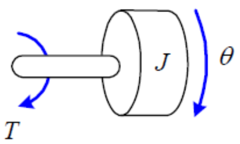
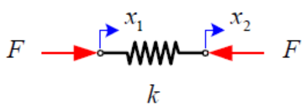
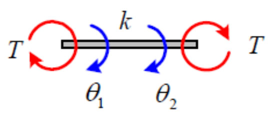
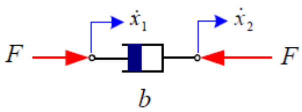
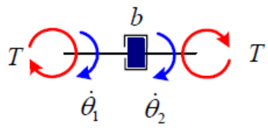
Where B relating the damping force F to the velocity and called the ***viscous friction coefficient.*** The dimension of b is [force/Velocity] = [N.s/m] in SI units.

For the torsional damper shown in Fig. 5, the torque T applied to the ends of the damper is:

$$T = B\,\dot{\theta}$$

Where B relating the damping torque T to the angular velocity and called the ***viscous friction coefficient.*** The dimension of B is [torque/angular velocity] = [N.m.s/rad] in SI units.



Element	Translation	Rotation
Inertia	 $\sum F = m\,a$	 $\sum T = J\,\alpha$
Spring	 $F = k(x_1 - x_2) = kx$	 $T = k(\theta_1 - \theta_2) = k\theta$
Damper	 $F = b(\dot{x}_1 - \dot{x}_2) = b\dot{x}$	 $T = b(\dot{\theta}_1 - \dot{\theta}_2) = b\dot{\theta}$

Example (1):

Write the differential equations describing systems shown in Fig. 6.

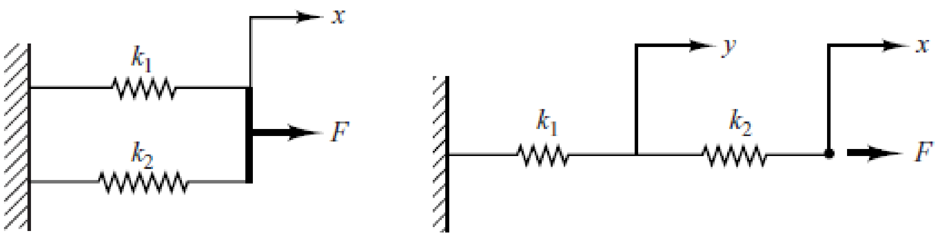


Fig. 6: a) parallel springs b) series springs

For system in Fig. 6. a)

$$F = k_1 x + k_2 x$$

For system in Fig. 6. b)

$$F = k_2(x - y) \qquad (at\ node\ x)$$

$$0 = k_1 y + k_2(y - x) \qquad (at\ node\ y)$$

Example (2):

For the mechanical system shown in Fig. 7, draw the mechanical network and write the D.E at each node.

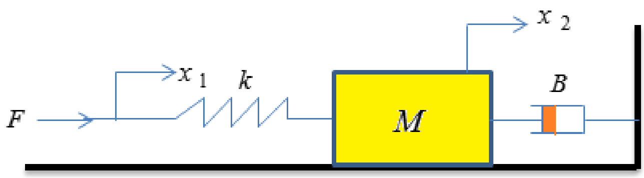
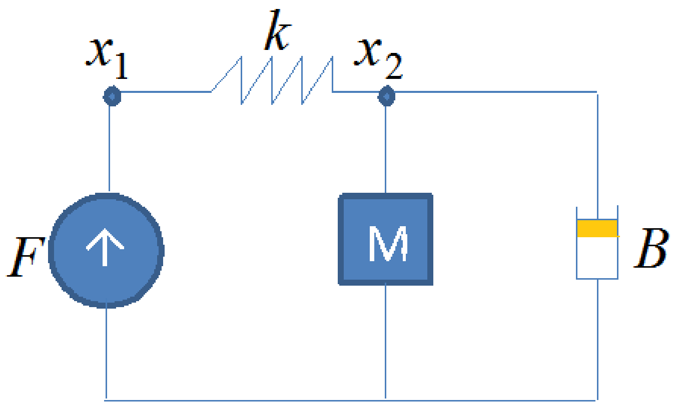


Fig. 7, One-mass mechanical system

The mechanical network is:



At node x_1 :

$f(t) = k (x_1 – x_2)$

$F(s) = k X_1(s) – k X_2(s)$

At node x_2 :

$$0 = k(x_2 - x_1) + M\ddot{x}_2 + B\dot{x}_2$$

$0 = k X_2(s) – k X_1(s) – M S^2X_2(s) + B SX_2(s)$

<u>Example (3):</u>

Obtain the transfer functions $X_1(s)/F(s)$ of the mechanical system shown in Fig. 8.

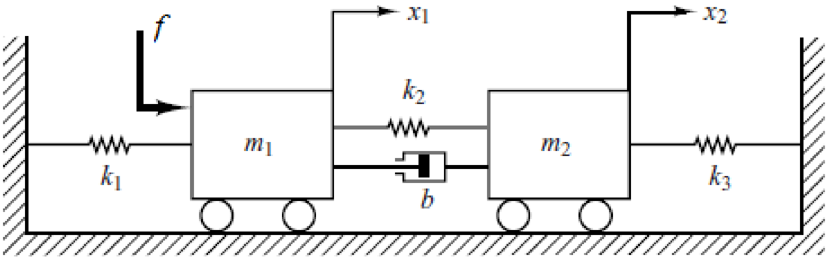
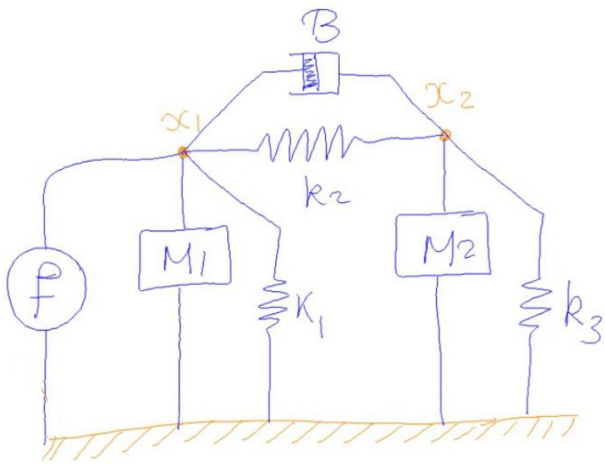


Fig. 8, Two-mass mechanical system



Mechanical network

Writing the D.E. at the displacement x_1:

$$f(t) = m_1\ddot{x}_1 + b(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2)$$

Taking Laplace:

$$F(s) = m_1S^2X_1(s) + bSX_1(s) - bSX_2(s) + k_1X_1(s) + k_2X_1(s) - k_2X_2(s)$$

$$F(s) = X_1(s)[m_1S^2 + bS + k_1 + k_2] - X_2(s)[bS + k_2] \quad (1)$$

Writing the D.E. at the displacement x_2:

$$0 = m_2\ddot{x}_2 + b(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + k_3x_2$$

Taking Laplace:

$$0 = m_2S^2X_2(s) + bSX_2(s) - bSX_1(s) + k_2X_2(s) + k_3X_2(s) - k_2X_1(s)$$

$$0 = X_2(s)[m_2S^2 + bS + k_2 + k_3] - X_1(s)[bS + k_2] \quad (2)$$

From Eqn. (2):

$$X_2(s) = \frac{bS + k_2}{m_2S^2 + bS + k_2 + k_3}X_1(s)$$

Substituting with the value of $X_2(s)$ in eqn. (1)

$$F(s) = X_1(s)[m_1S^2 + bS + k_1 + k_2] - \frac{(bS + k_2)^2}{m_2S^2 + bS + k_2 + k_3}X_1(s)$$

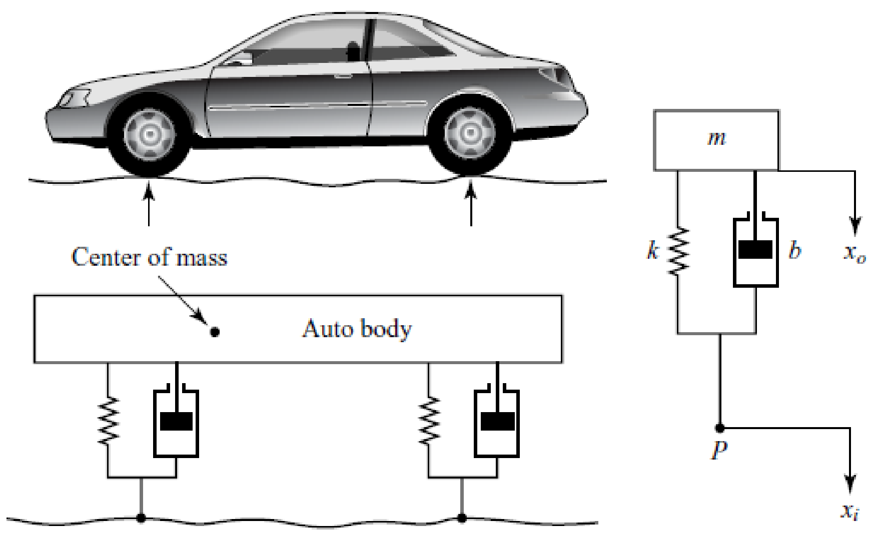
$$F(s) = \frac{(m_1S^2 + bS + k_1 + k_2)(m_2S^2 + bS + k_2 + k_3) - (bS + k_2)^2}{m_2S^2 + bS + k_2 + k_3}X_1(s)$$

Then

$$\frac{X_1(s)}{F(s)} = \frac{m_2S^2 + bS + k_2 + k_3}{(m_1S^2 + bS + k_1 + k_2)(m_2S^2 + bS + k_2 + k_3) - (bS + k_2)^2}$$

<u>Example (4):</u>

For a car suspension shown in Fig. 8,



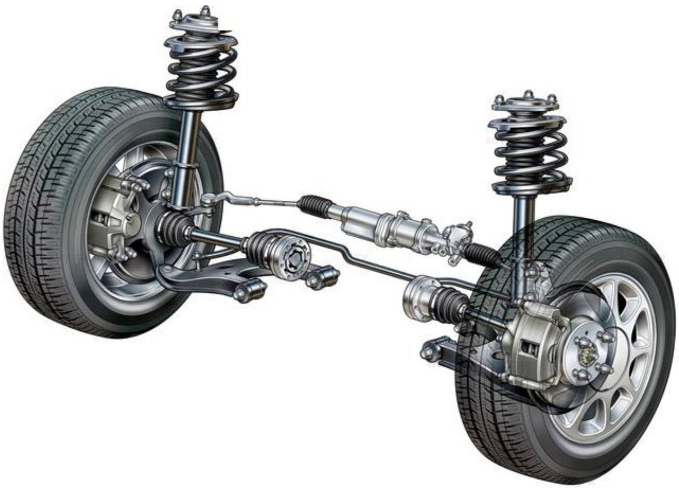


Fig.8, Car suspension system

The equation of motion for the suspension system is:

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

That can be rewrite as:

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i$$

Taking Laplace:

$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Then the system T.F. is:

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Example (5):

For the mechanical system shown in Fig. 9, write the differential equation at each displacement then find the dynamic equation of that system. Consider x_2 as output.

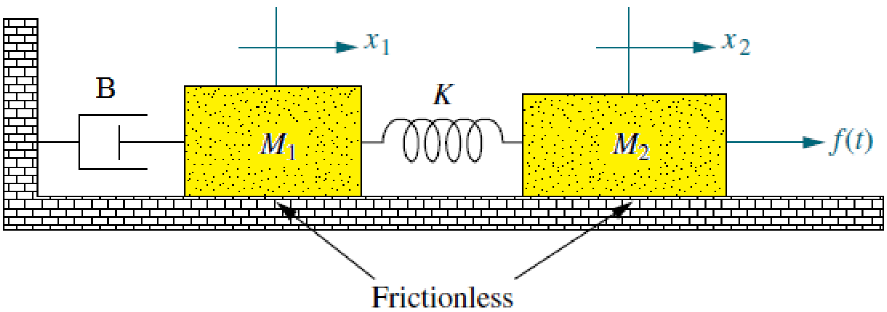


Fig. 9, Two-mass mechanical system

Let v_1, x_1, v_2 and x_2 are the state variables

We know that:

$$\frac{dx_1}{dt} = v_1 \quad and \quad \frac{dx_2}{dt} = v_2$$

Writing the D.E. at the displacement x_2:

$$f(t) = M_2\ddot{x}_2 + k(x_2 - x_1)$$

$$f(t) = M_2\dot{v}_2 + k(x_2 - x_1)$$

$$\dot{v}_2 = \frac{1}{M_2}f(t) - \frac{k}{M_2}x_2 + \frac{k}{M_2}x_1$$

Writing the D.E. at the displacement x_1:

$$0 = M_1\ddot{x}_1 + B\dot{x}_1 + k(x_1 - x_2)$$

$$\dot{v}_1 = -\frac{B}{M_1}v_1 - \frac{k}{M_1}x_1 + \frac{k}{M_1}x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M_1} & -\frac{B}{M_1} & \frac{k}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{M_2} & 0 & -\frac{k}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} u(t)$$

$$[y] = [0 \quad 0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

Example (6):

Find the T.F. $\theta_2(s)/T(s)$ for the rotational mechanical system shown in Fig. 10.

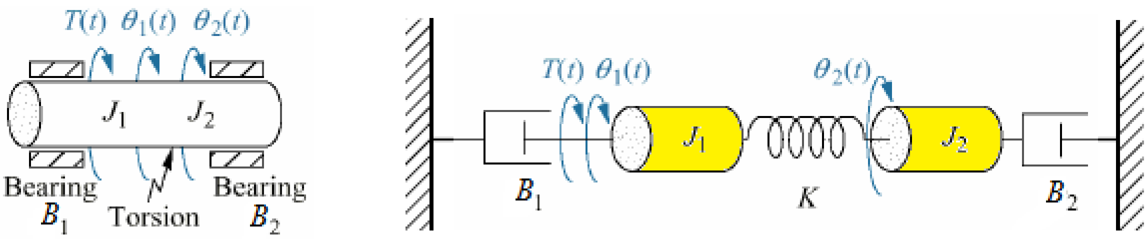


Fig. 10, Rotational mechanical system

Writing the D.E. at the angular displacement θ_1:

$$T(t) = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + k(\theta_1 - \theta_2)$$

Taking Laplace:

$$T(s) = J_1S^2\theta_1(s) + B_1S\theta_1(s) + k\theta_1(s) - k\theta_2(s)$$

$$T(s) = \theta_1(s)[J_1S^2 + B_1S + k] - k\,\theta_2(s) \qquad (1)$$

Writing the D.E. at the angular displacement θ_2:

$$0 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + k(\theta_2 - \theta_1)$$

Taking Laplace:

$$0 = J_2S^2\theta_2(s) + B_2S\theta_2(s) + k\theta_2(s) - k\theta_1(s)$$

$$0 = \theta_2(s)[J_2S^2 + B_2S + k] - k\,\theta_1(s) \qquad (2)$$

From Eqn. (2):

$$\theta_1(s) = \frac{[J_2S^2 + B_2S + k]}{k}\theta_2(s)$$

Substituting with the value of $\theta_2(s)$ in eqn. (1),

$$T(s) = \frac{[J_2S^2 + B_2S + k]}{k}[J_1S^2 + B_1S + k]\theta_2(s) - k\,\theta_2(s)$$

$$T(s) = \frac{[J_2S^2 + B_2S + k][J_1S^2 + B_1S + k] - k^2}{k}\theta_2(s)$$

Then the system T.F. is:

$$\frac{\theta_2(s)}{T(s)} = \frac{k}{[J_2S^2 + B_2S + k][J_1S^2 + B_1S + k] - k^2}$$

Example (7):

Write the D.E's describe the rotational mechanical system shown in Fig. 11. Then draw the block diagram and calculate the T.F. $\theta_2(s)/T(s)$.

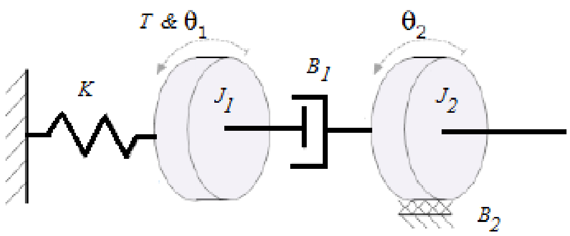
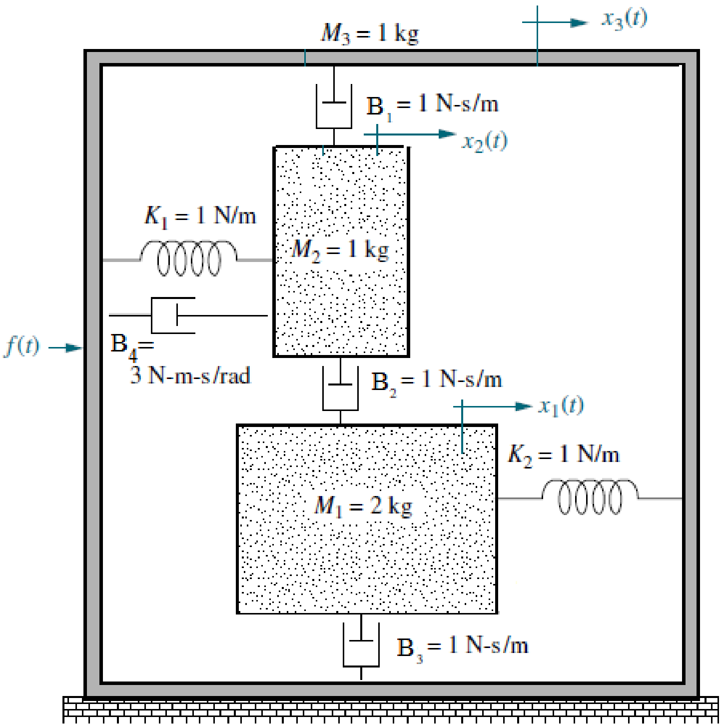


Fig. 11, Rotational system

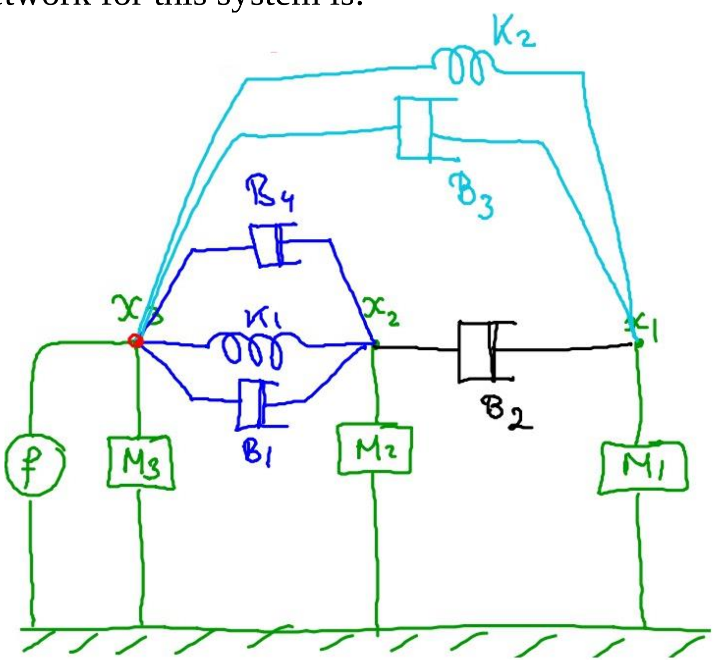
Solution at smart board lecture.

Example (8):

For the linear displacement mechanical system shown below, draw the mechanical network, then write the D.E's that describe the system and draw the block diagram where $x_1(t)$ is the desired output

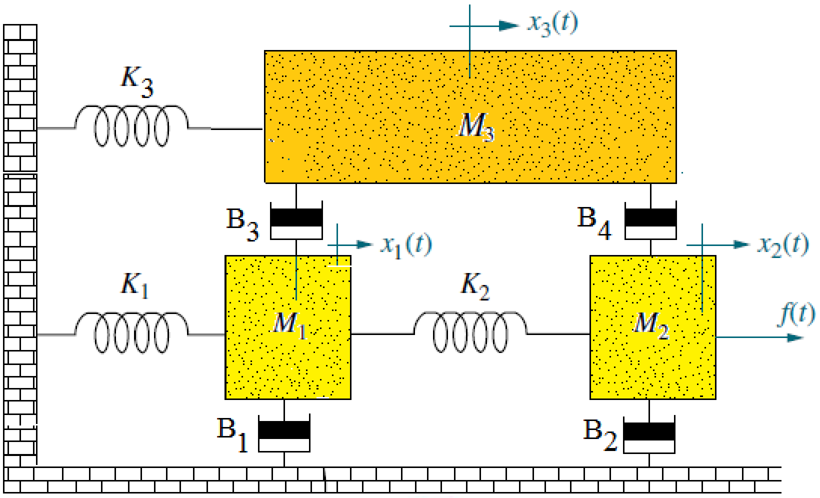


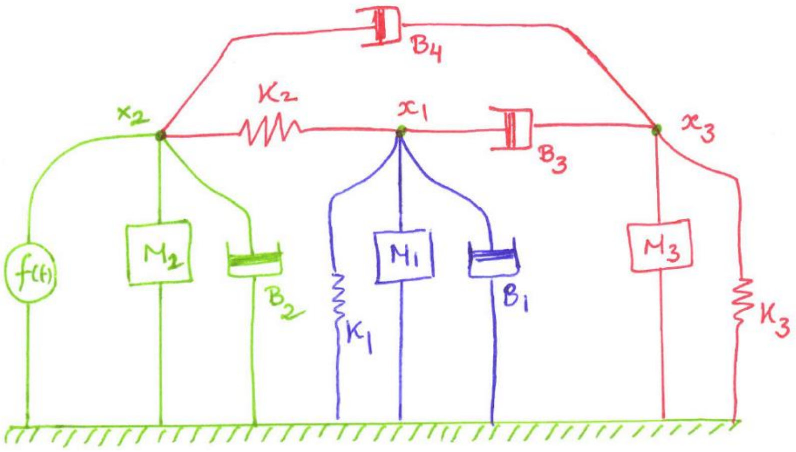
The mechanical network for this system is:



Example (9):

For the translational mechanical system shown below, draw the mechanical network, then write the system differential equations and draw the block diagram. (consider x_3 as output)





at node x_2:

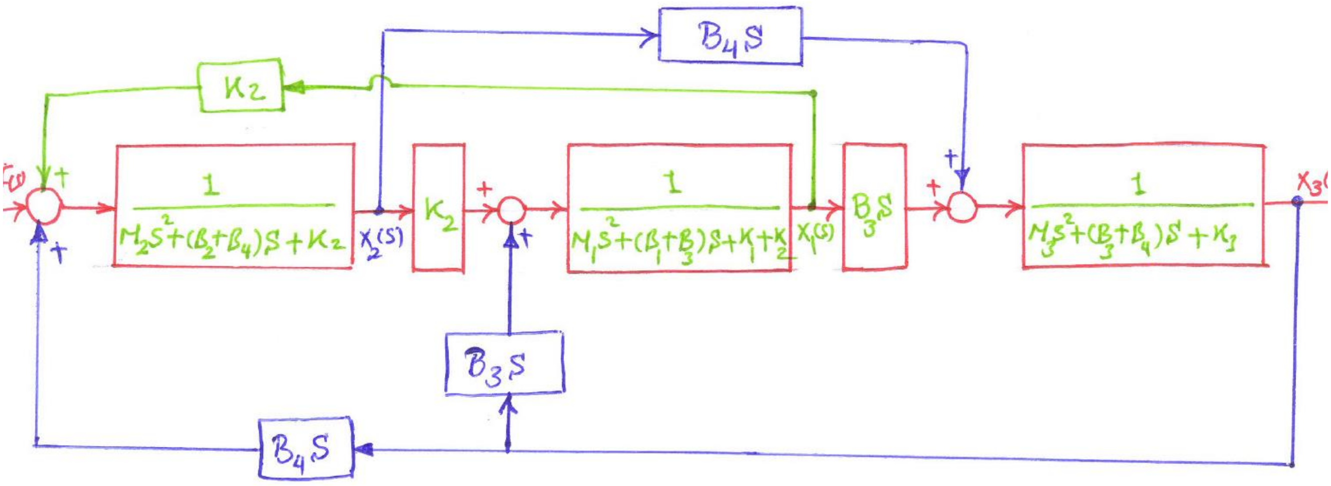
$$F(s) = X_2(s)[M_2 s^2 + B_2 s + B_4 s + K_2] - X_1(s) K_2 - X_3(s) B_4 s \quad \text{——} \; (1)$$

at node X_1:

$$0 = X_1(s)[M_1 s^2 + (B_1 + B_3)s + K_1 + K_2] - X_2(s) K_2 - X_3(s) B_3 s \quad \text{——} \; (2)$$

at node X_3:

$$0 = X_3(s)[M_3 s^2 + (B_3 + B_4)s + K_3] - X_1(s) B_3 s - X_2(s) B_4 s \quad \text{——} \; (3)$$



5. Mechanical Systems with Gears

Gear is a toothed machine part, such as a wheel or cylinder that meshes with another toothed part to transmit motion or to change speed or direction.

In industrial applications, generally gears associate to a motor which drives the load. Gears are used to obtain more speed and less torque or less speed and more torque. The interaction between two gears is depicted in the Fig. 12. An input gear with radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius r2 and N2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$.

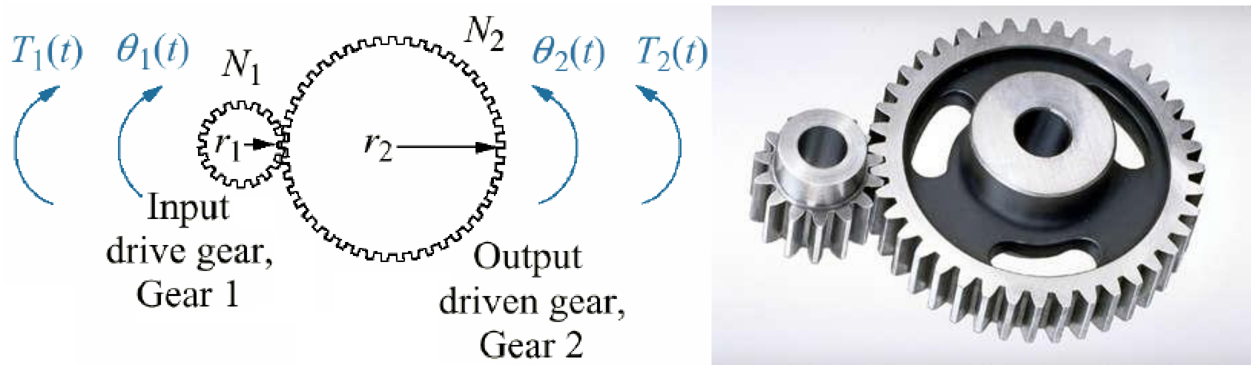


Fig. 12, Two-Gear transmission system

Also we must note that, if the number of gear is even, the direction of motion is reversed. But if it is odd, as shown in Fig. 13, the direction of motion is not reversed.

Fig. 13, Three-Gear transmission system

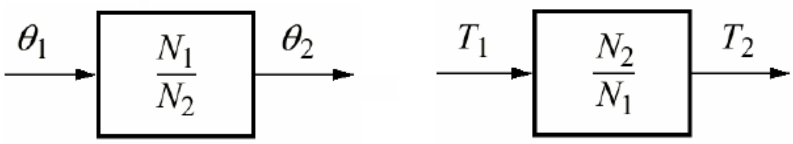
What is the relationship between the input torque, T_1 and the delivered torque, T_2 ? Assuming the gears do not absorb or store energy (ideal gear), then the input energy of Gear 1 equals the energy out of Gear 2.

$$T_1 \times \theta_1 = T_2 \times \theta_2$$

Therefore,

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$$

These relations can be summarized in blocks as:



Example (10):

For the gear train shown in Fig. 14, a load is driven by a motor through the gear train. Assuming the stiffness of the motor shaft is infinite, draw the block diagram and find the T.F. $\theta_2(s)/T_m(s)$.

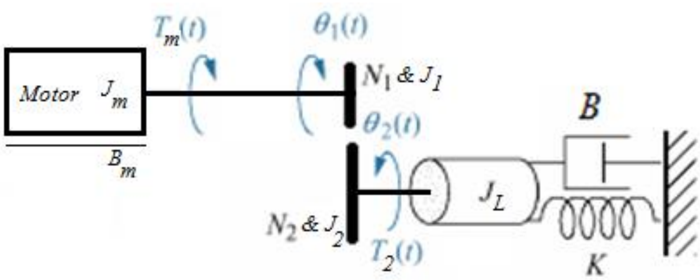


Fig. 14, Gear train system

At node θ_1:

$$T_m(t) = (J_m + J_1)\ddot{\theta}_1 + B_m\dot{\theta}_1 + T_1(t)$$

$$T_m(s) = \theta_1(s)[\,(J_m + J_1)S^2 + B_m S\,] + T_1(s) \qquad (1)$$

At node θ_2:

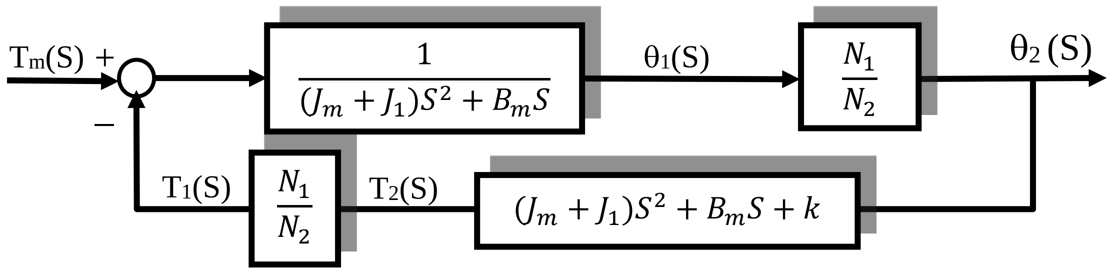
$$T_2(t) = (J_2 + J_L)\ddot{\theta}_2 + B\left(\dot{\theta}_2\right) + k\,\theta_2$$

$$T_2(s) = \theta_2(s)[\,(J_2 + J_L)S^2 + BS + k] \qquad (2)$$

Also we must consider the two relations of the gear train:

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$$

From the above eqns., we can draw the block diagram:

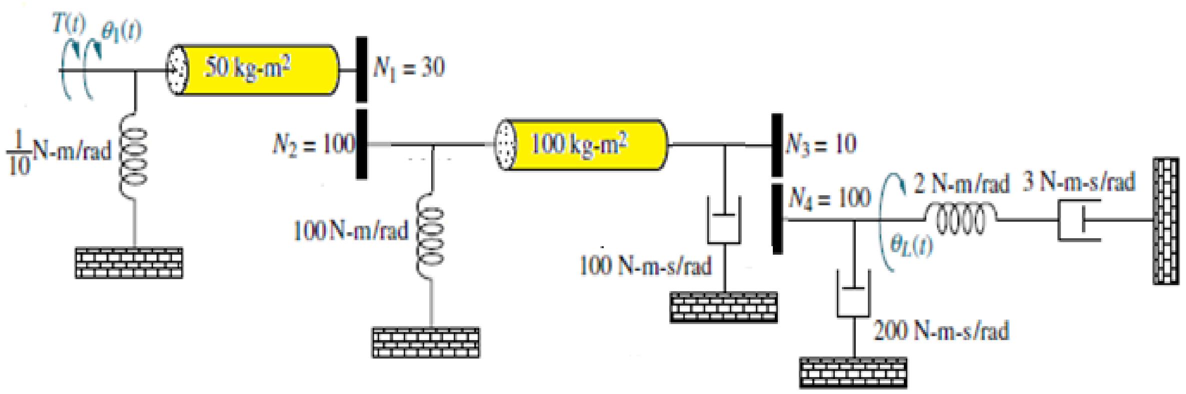


So you can easily calculate the system T.F. $\theta_2(s)/T_m(s)$

<u>Example (11):</u>

For the rotational mechanical system given below,

a) Write the differential equations that represent that system,
b) Draw the block diagram considering T(s) as input and $\theta_L(s)$ as an output.



The D.E's that describe the mechanical system are:

$T(s) = \theta_1(s)[50\,S^2 + 0.1] + T_1(s)$

$T_2(s) = \theta_2(s)[100\ S^2 + 100\ S + 100] + T_3(s)$

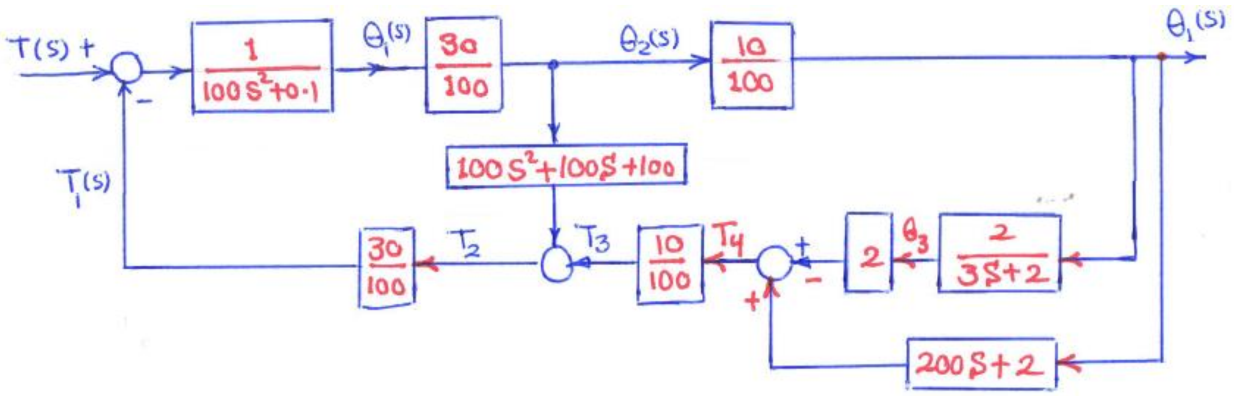
$T_4(s) = \theta_L(s)[200\ S + 2] - \theta_3(s)[2]$

$0 = \theta_3(s)[3\ S + 2] - \theta_L(s)[2]$

$$\frac{N_1}{N_2} = \frac{30}{100} = \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1}$$

$$\frac{N_3}{N_4} = \frac{10}{100} = \frac{T_3}{T_4} = \frac{\theta_L}{\theta_2}$$

The above equations can be represented in block diagram as:

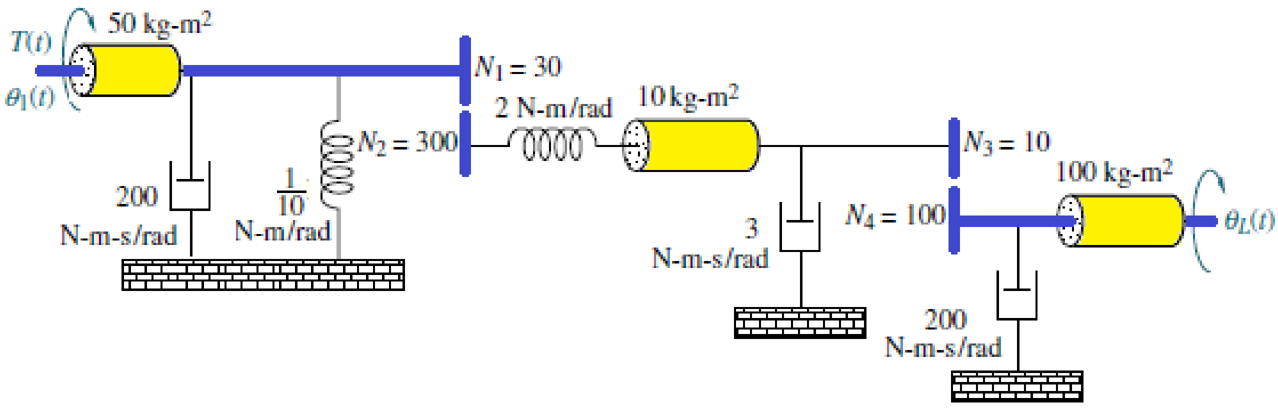


Example (12):

For the rotational mechanical system given below,

a) Write the differential equations that represent that system,

b) Draw the block diagram considering T(s) as input and $\theta_L(s)$ as an output.



The D.E's that describe the mechanical system are:

$T(s) = \theta_1(s)[50\ S^2 + 200S + 0.1] + T_1(s)$

$T_2(s) = \theta_2(s)[2] - \theta_3(s)[2]$

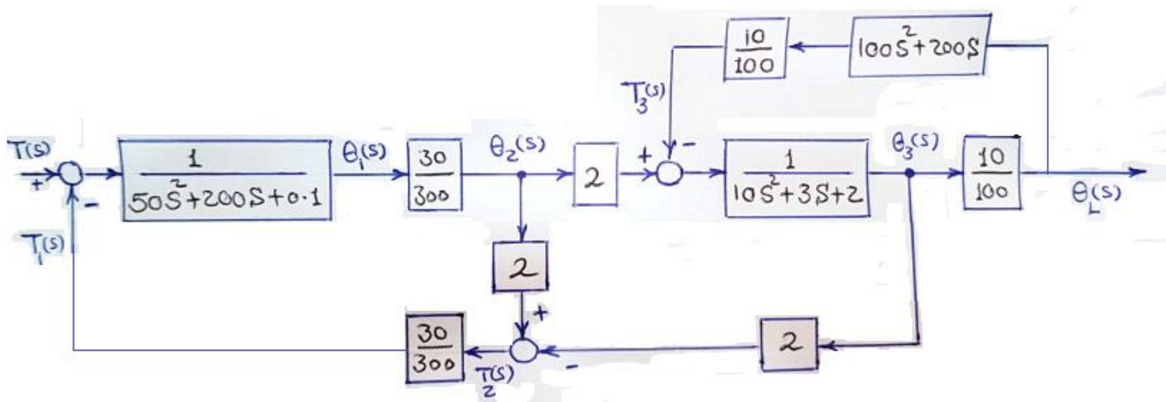
$0 = \theta_3(s)[\ 10\ S^2 + 3S + 2] - \theta_2(s)[2] + T_3(s)$

$T_4(s) = \theta_L(s)[\ 100\ S^2 + 200S]$

$$\frac{N_1}{N_2} = \frac{30}{300} = \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1}$$

$$\frac{N_3}{N_4} = \frac{10}{100} = \frac{T_3}{T_4} = \frac{\theta_L}{\theta_3}$$

The above equations can be represented in block diagram as:



6. Modeling of Electrical Systems

A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

***RC* Circuit:** Consider the electrical circuit shown in Fig. 15. The circuit consists of a resistance R (ohm), and a capacitance C (farad).

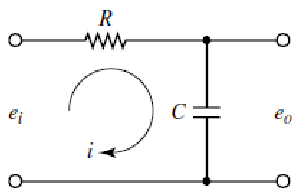
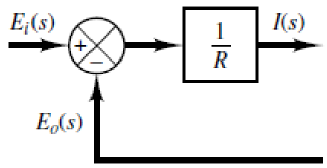


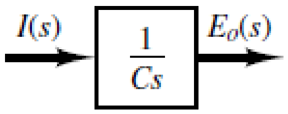
Fig. 15. RC circuit

The equations of this RC circuit are:

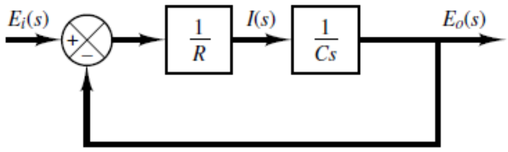


$$i = \frac{e_i - e_o}{R} \quad \Rightarrow \quad I(s) = \frac{E_i(s) - E_o(s)}{R}$$

$$e_o = \frac{\int i\, dt}{C} \quad \Rightarrow \quad E_o(s) = \frac{I(s)}{Cs}$$



Combining the above two blocks we get the overall block diagram of the RC circuit;



$$\frac{E_o(s)}{E_i(s)} = \frac{1}{1 + RCs}$$

***RLC* Circuit:** Consider the electrical circuit shown in Fig. 16. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad).

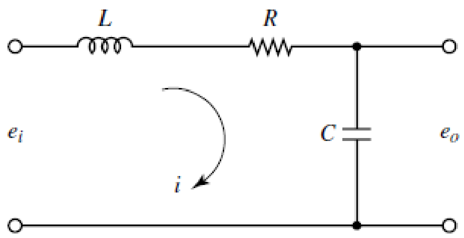


Fig. 16, RLC circuit

Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

$$\frac{1}{C}\int i\,dt = e_o$$

Taking Laplace:

$$E_i(s) = I(s)\left\{Ls + R + \frac{1}{Cs}\right\} = \frac{LCs^2 + RCs + 1}{Cs} I(s)$$

$$E_o(s) = I(s)\{\frac{1}{Cs}\}$$

The block diagram is given below:

From which the T.F. is:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

A state-space model of that system may be obtained as follows:

First, note that the differential equation for the system can be obtained from T.F. as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Assuming the state variables as:

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Repeated RC circuit: as shown in Fig. 17, we need to obtain the T.F. of this circuit. Therefore the D.E's. that describe the circuit are as follows:

$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i \longrightarrow \frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \longrightarrow \frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o \longrightarrow \frac{1}{C_2 s} I_2(s) = E_o(s)$$

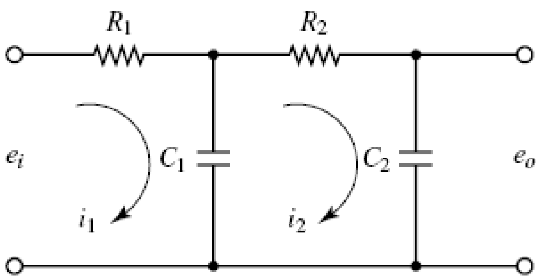
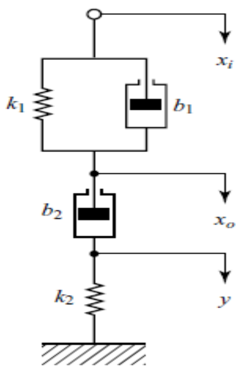


Fig. 17. Cascaded RC circuit

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s}$$
$$= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$

Example (13):

Obtain the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Fig. 18 (a). Also obtain the transfer function $E_o(s)/E_i(s)$ of the electrical system shown in Fig. 18 (b). Show that these transfer functions of the two systems are of identical form and thus they are analogous systems.



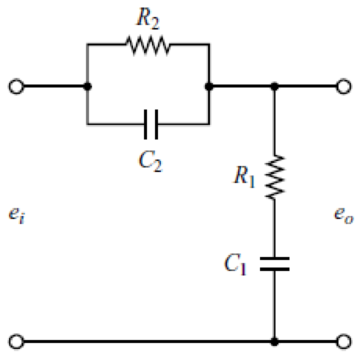


Fig. 18, (a) Mechanical system (b) analogous electrical system

From mechanical system:

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$

$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

Taking Laplace:

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)]$$

$$b_2[sX_o(s) - sY(s)] = k_2 Y(s)$$

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2 sX_o(s) - b_2 s\frac{b_2 sX_o(s)}{b_2 s + k_2}$$

or

$$(b_1 s + k_1)X_i(s) = \left(b_1 s + k_1 + b_2 s - b_2 s\frac{b_2 s}{b_2 s + k_2}\right)X_o(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ can be obtained as

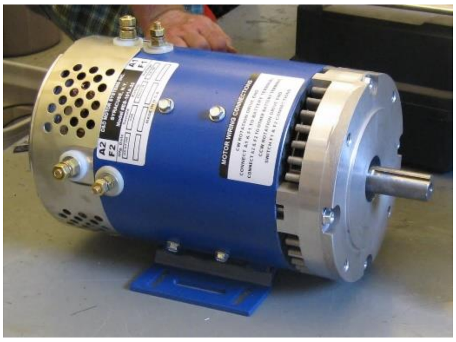
$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

From the electrical system:

$$\frac{E_o(s)}{E_i(s)} = \frac{R_1 + \frac{1}{C_1 s}}{\frac{1}{(1/R_2) + C_2 s} + R_1 + \frac{1}{C_1 s}} = \frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_2 C_1 s}$$

7. Modeling of DC Machines:

Direct-current (DC) motors are one of the most widely used prime movers in the industry. Years ago, the majority of the small servomotors used for control purposes were ac. In reality, ac motors are more difficult to control, especially for position control, and their characteristics are quite nonlinear, which makes the analytical task more difficult. DC motors, on the other hand, are more expensive, because of their brushes and commutators, and variable-flux DC motors are suitable only for certain types of control applications. Before permanent-magnet technology was fully developed, the torque-per-unit volume or weight of a DC motor with a permanent-magnet (PM) field was far from desirable. Today, with the development of the rare-earth magnet, it is possible to achieve very high torque-to-volume PM DC motors at reasonable cost. Furthermore, the advances made in brush-and-commutator technology have made these wearable parts practically maintenance-free. The advancements made in power electronics have made brushless dc motors quite popular in high-performance control systems. Advanced manufacturing techniques have also produced dc motors with ironless rotors that have very low inertia, thus achieving a very high torque-to-inertia ratio. Low-time-constant properties have opened new applications for dc motors in computer peripheral equipment such as tape drives, printers, disk drives, and word processors, as well as in the automation and machine-tool industries.



The dc motor is basically a torque transducer that converts electric energy into mechanical energy. It consists from **<u>Stator</u>** that contain the field flux and **<u>Rotor</u>** (armature) that contains the windings. DC motor is modeled as a circuit with resistance R_a connected in series with an inductance L_a, and a voltage source e_b, representing the back emf (electromotive force) in the armature when the rotor rotates as shown in Fig. 19.

The torque developed (T_m) on the motor shaft is directly proportional to the field flux (ϕ) and the armature current (I_a).

$$T_m(t) = k\emptyset\, i_a(t)$$

If the flux is kept constant

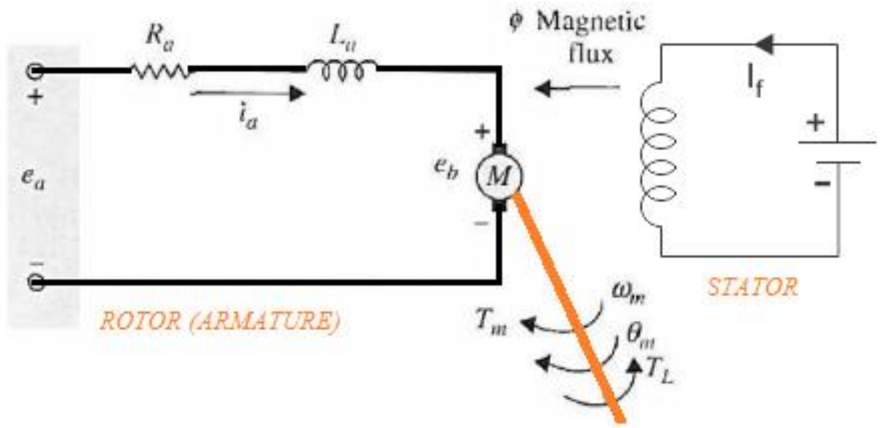
$$T_m(t) = k_i i_a(t)$$

Also the induced emf e_b is directly proportional to the field flux (ϕ) and the shaft speed (ω_m).

$$e_b(t) = k\emptyset\, \omega_m(t)$$

If the flux is kept constant

$$e_b(t) = k_b\, \omega_m(t)$$



$i_a(t)$ = armature current	L_a = armature inductance
R_a = armature resistance	$e_a(t)$ = applied voltage
$e_b(t)$ = back emf	K_b = back-emf constant
$T_L(t)$ = load torque	ϕ = magnetic flux in the air gap
$T_m(t)$ = motor torque	$\omega_m(t)$ = rotor angular velocity
$\theta_m(t)$ = rotor displacement	J_m = rotor inertia
K_i = torque constant	B_m = viscous-friction coefficient

Fig. 19, Separately-Excited DC motor circuit

Electrical Equation:

$$e_a(t) = R_a i_a(t) + L_a \frac{di_a}{dt} + e_b(t)$$

$$\frac{di_a}{dt} = \frac{1}{L_a} e_a(t) - \frac{R_a}{L_a} i_a(t) - \frac{1}{L_a} e_b(t)$$

<u>Mechanical Equation:</u>

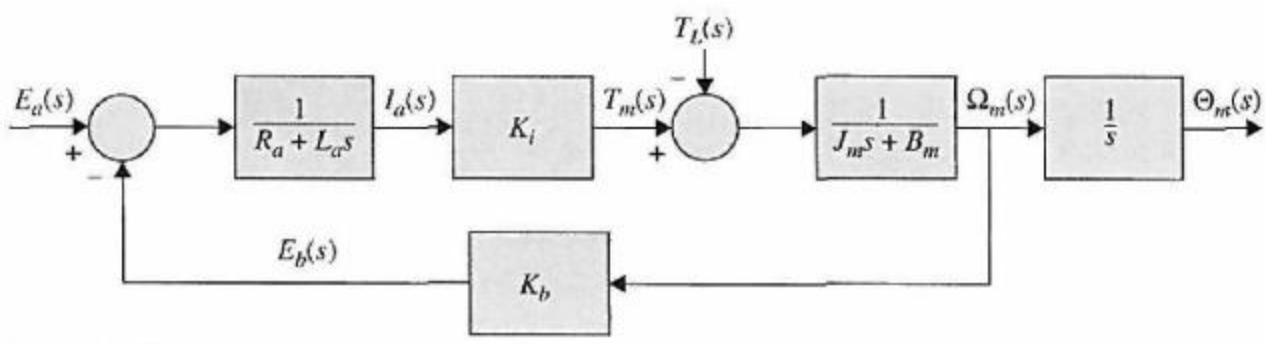
$$T_m(t) = J_m\ddot{\theta}_m(t) + B_m\dot{\theta}_m(t) + T_L$$

$$\ddot{\theta}_m(t) = \frac{1}{J_m}T_m(t) - \frac{B_m}{J_m}\dot{\theta}_m(t) - \frac{1}{J_m}T_L$$

The state variables of the system can be defined as $i_a(t)$, $\omega_m(t)$, and $\theta_m(t)$. The state equations of the dc-motor system are written in vector-matrix form:

$$\begin{bmatrix} \frac{di_a(t)}{dt} \\ \frac{d\omega_m(t)}{dt} \\ \frac{d\theta_m(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} & 0 \\ \frac{K_i}{J_m} & -\frac{B_m}{J_m} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a(t) \\ \omega_m(t) \\ \theta_m(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} e_a(t) + \begin{bmatrix} 0 \\ -\frac{1}{J_m} \\ 0 \end{bmatrix} T_L(t)$$

The block diagram of dc motor is given below:



At steady-state, the term di/dt is zero, and Eqn (1) can be rewritten as:

$$e_a(t) = R_a i_a(t) + e_b(t)$$

$$e_a(t) = R_a \frac{T_m(t)}{k_t} + k_b\,\omega_m(t)$$

$$\omega_m(t) = \frac{e_a(t)}{k_b} - \frac{R_a}{k_b k_t}T_m(t)$$

The above equation represents the torque-speed characteristic of separately-excited DC motor and shown in figure below.

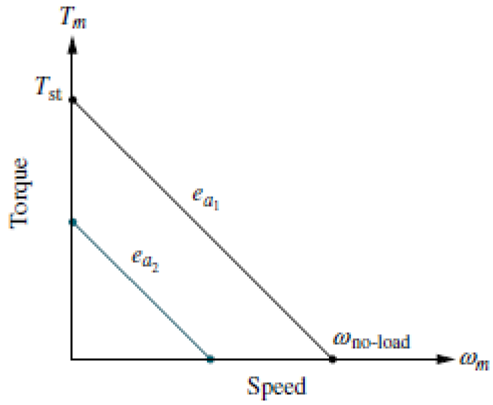
From this characteristic, at starting: ω_m=0 and $T_m = T_{st}$

$$e_a(t) = R_a\frac{T_{st}}{k_t} \;\rightarrow\; k_t = R_a\frac{T_{st}}{e_a(t)}$$

At no load, the speed is no-load speed (ω_{nL}) and the torque is zero, $T_m = 0$

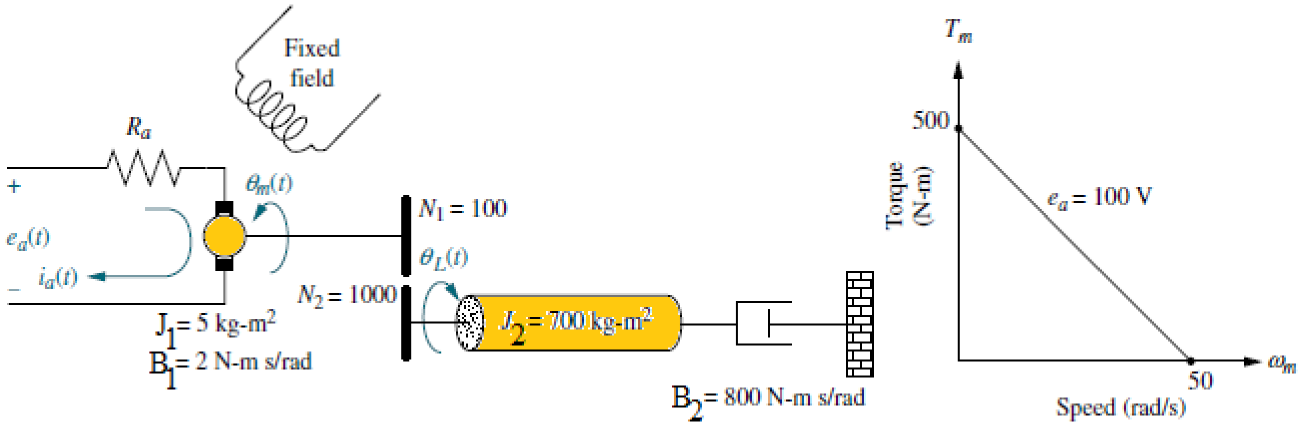
$$\omega_{nL}(t) = \frac{e_a(t)}{k_b} \rightarrow k_b = \frac{e_a(t)}{\omega_{nL}}$$

From the above two equations, the electrical constants k_t & k_b can be determined.



Example (13)

For the separately-excited DC motor with torque-speed characteristic given below, draw the block diagram then find the transfer function $\theta_L(S)/E_a(S)$. Take the armature resistance and inductance are 0.06 Ω and 0.2 H respectively.



From the torque-Speed characteristic above, ω_{nL}=50 rad/s, T_{st}=500 N.m at E_a 100V. Therefore, the motor constants can be obtained as:

$$k_t = R_a \frac{T_{st}}{e_a(t)} = \frac{0.06 \times 500}{100} = 0.3\ N.m/A$$

$$k_b = \frac{e_a(t)}{\omega_{nL}} = \frac{100}{50} = 2\ V.s/rad$$

Electrical equation at constant flux (S-domain):

$$E_a(S) = I_a(S)[R_a + L_a S] + k_b \omega(S)$$

$$E_a(S) = I_a(S)[0.06 + 0.2\, S] + 2S\, \theta(S)$$

The electromagnetic torque $T_m(S) = k_t\, I_a(S) = 0.3\, I_a(S)$

<u>Mechanical equation at constant flux (S-domain):</u>

$$T_m(S) = \theta_m(S)[J_1 S^2 + B_1 S] + T_1$$

$$T_m(S) = \theta_m(S)[5S^2 + 2S] + T_1$$

From the gear ratio:

$$\frac{\theta_L(S)}{\theta_m(S)} = \frac{N_1}{N_2} = \frac{100}{1000} = \frac{1}{10}$$

At load:

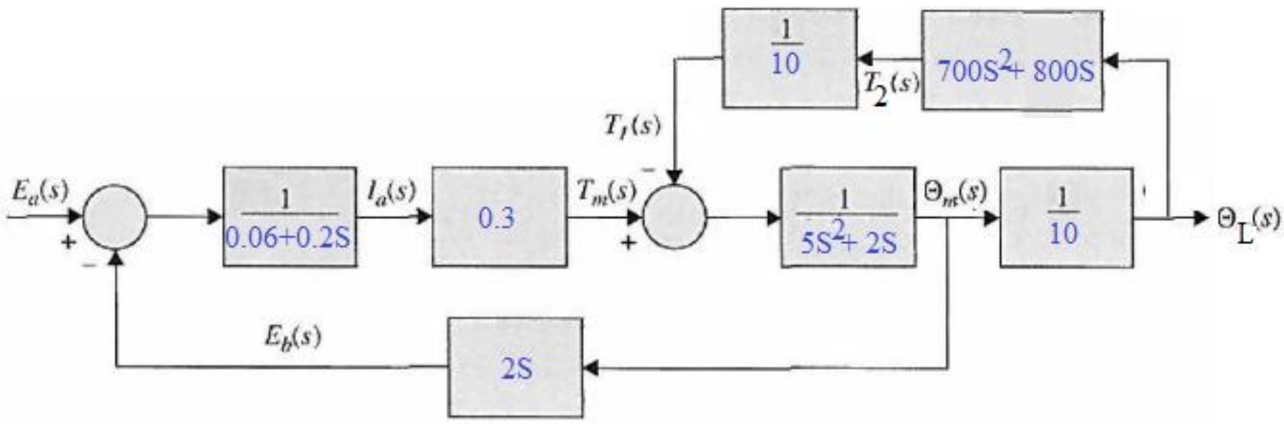
$$T_2(S) = \theta_L(S)[J_2 S^2 + B_2 S]$$

$$T_2(S) = \theta_L(S)[700\, S^2 + 800\, S]$$

From the gear ratio:

$$\frac{T_1(S)}{T_2(S)} = \frac{N_1}{N_2} = \frac{100}{1000} = \frac{1}{10}$$

The block diagram is given in figure below.



<u>Example (15):</u>

Consider the speed control system shown in Fig. 20. The armature of the motor is supplied with a controlled voltage through a DC generator. The generator field current controls the generated voltage E_g. Draw the block diagram representing this system and deduce the T.F. $\omega_m(s)/E_i(s)$

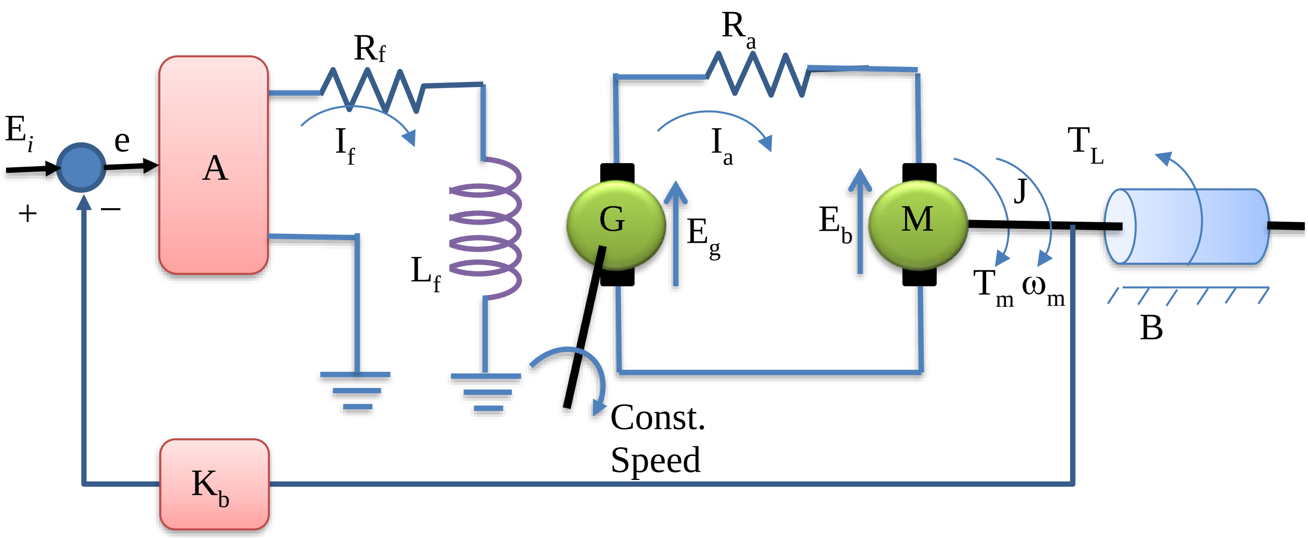


Fig. 20, Motor-Generator system

The D.E's that describe the motor-generator set are:

$$E_i(s) - k_b \omega_m(s) = e(s)$$

$$A\ e(s) = I_f(s)[R_f + SL_f]$$

$$E_g(s) = k_g I_f(s)$$

$$E_g(s) = R_a I_a(s) + E_b(s)$$

$$E_b(s) = k_m \omega_m(s)$$

$$T_m(s) = k_m I_a(s)$$

$$T_m(s) = [JS + B]\omega_m(s) + T_L(s)$$

By representing the above D.E's we can draw the block diagram: (refer to smart-board lrcture.

Example (16):

The mechanical system shown in Fig. 21, is used to measuring the displacement x_2 due to the driving force f(t). Write the D.E's describing this system, then draw the corresponding block diagram.

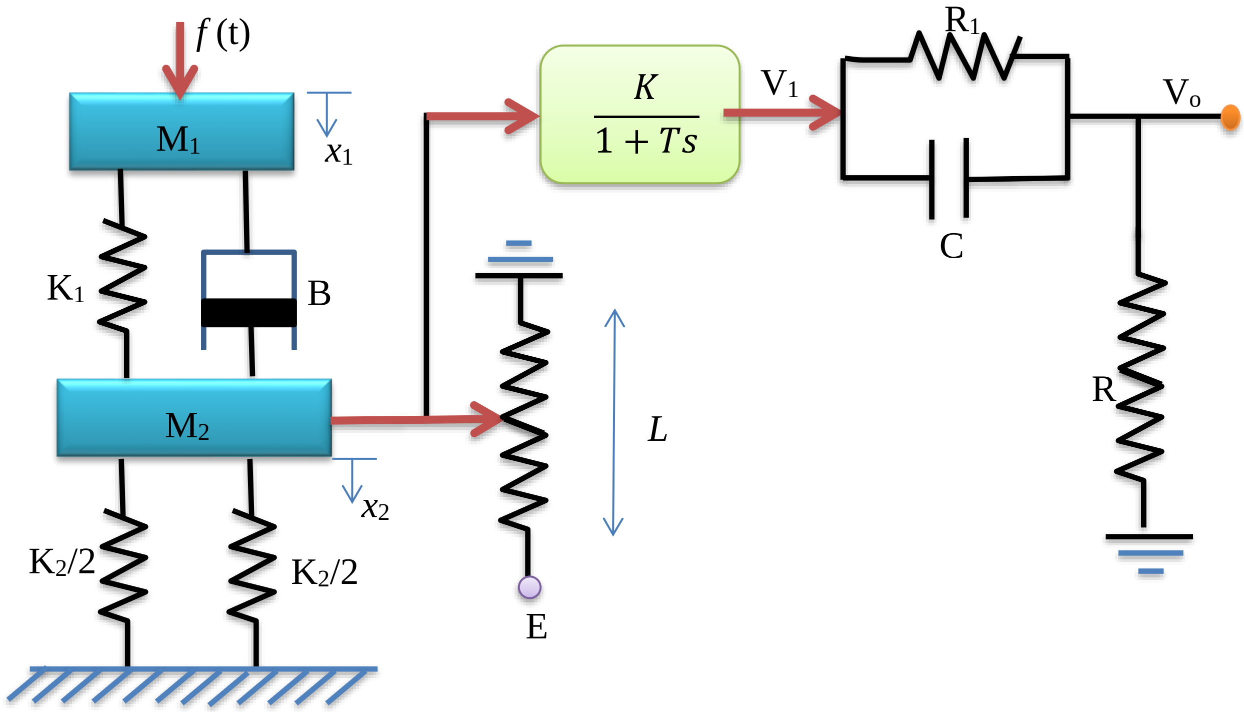
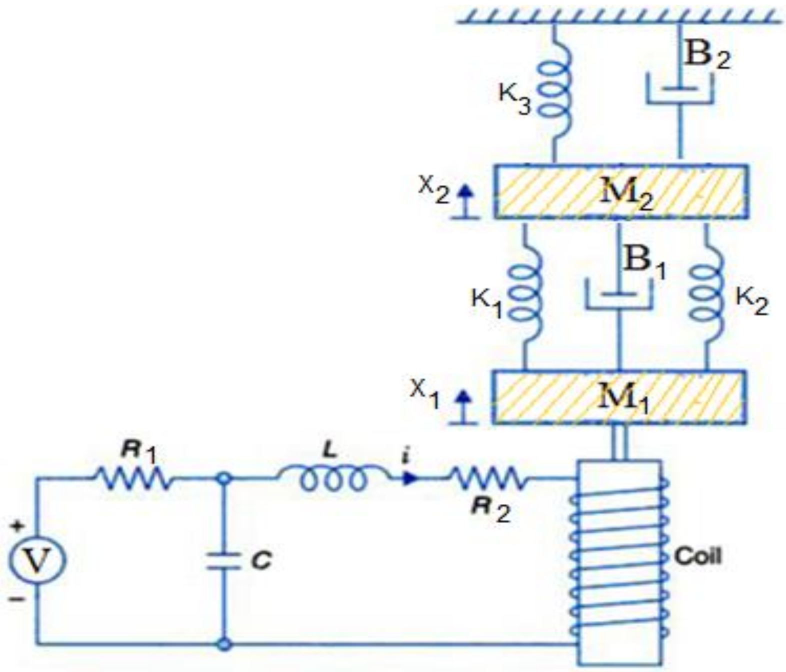


Fig. 21, Distance-detector system

Solution at the smart-board lectures.

Example (17):

For the electro-mechanical system shown below, the solenoid produces a magnetic force $F_C = K_C\, i$. Draw the block diagram then find $X_2(s)/V(s)$



for loop current i_1:

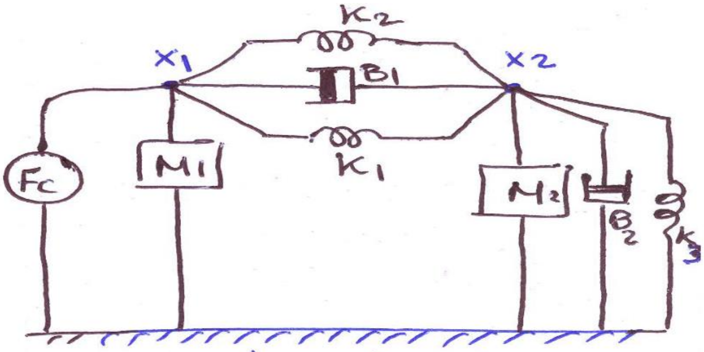
$$V(S) = I_1(S)\left[R_1 + \frac{1}{CS}\right] - I(S)\frac{1}{CS} \quad \text{—①}$$

$\left(R_1 + \frac{1}{CS} = \frac{R_1CS+1}{CS}\right)$

for loop current i_2:

$$0 = I(S)\left[R_2 + LS + \frac{1}{CS}\right] - I_1\frac{1}{CS} \quad \text{—②}$$

$$F_c(S) = K_c\, I(S) \quad \text{—③}$$



Mech. Network

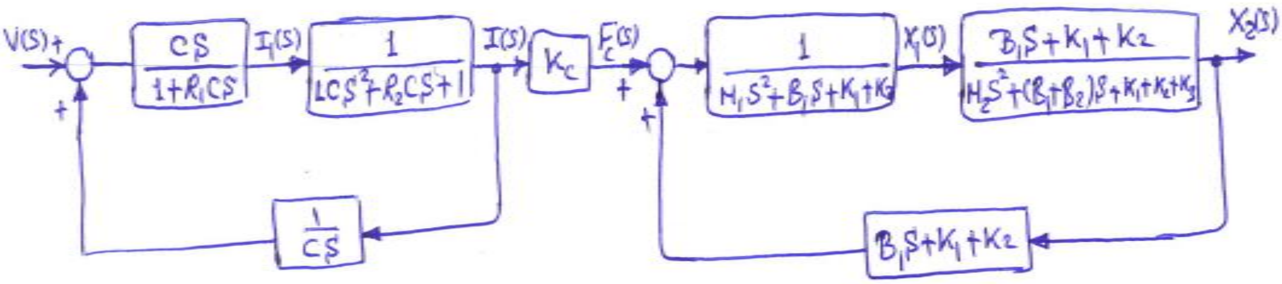
from Mechanical network

at node x_1:

$$F_c(S) = X_1(S)\left[M_1S^2 + B_1S + K_1 + K_2\right] - X_2(S)\left[B_1S + K_1 + K_2\right] \quad \text{—④}$$

at node x_2

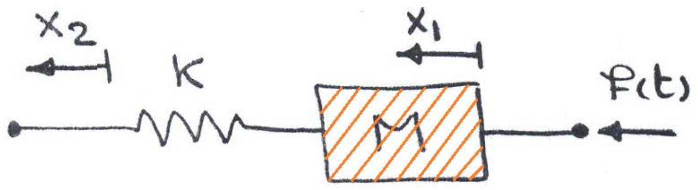
$$0 = X_2(S)\left[M_2S^2 + (B_1+B_2)S + K_1 + K_2 + K_3\right] - X_1(S)\left[B_1S + K_1 + K_2\right] \quad \text{—⑤}$$

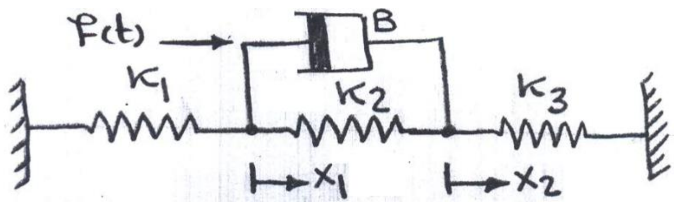


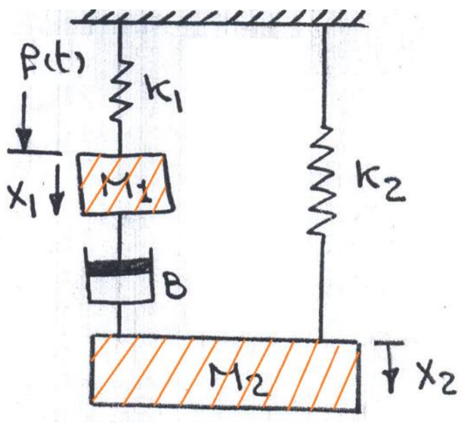
Sheet 5 (Physical Systems)

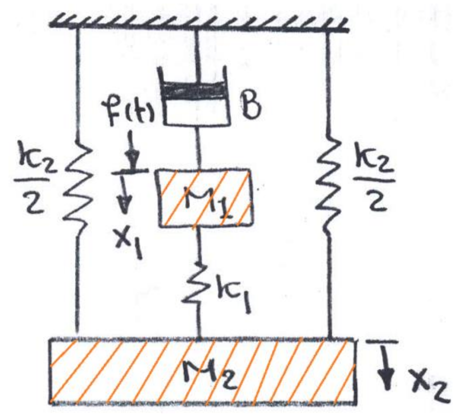
1) For the mechanical systems shown below;

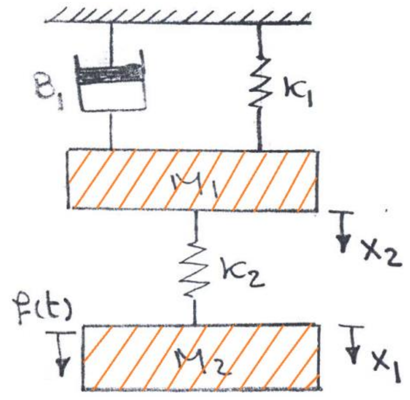
- Draw the mechanical network, then write the system D.E's
- If $X_2(s)$ is the system output, draw the block diagram and find $X_2(s)/F(s)$

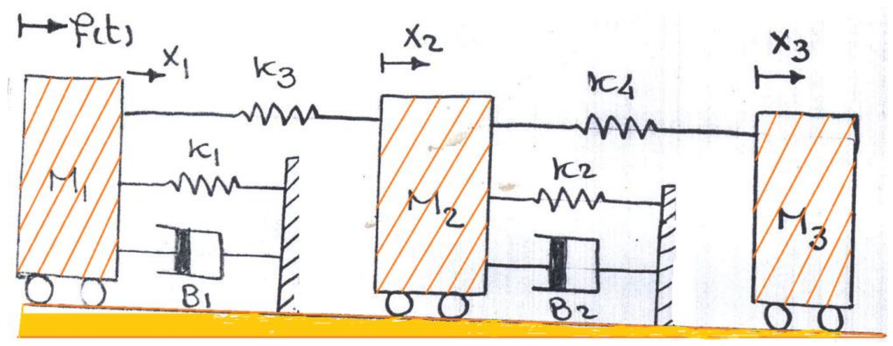


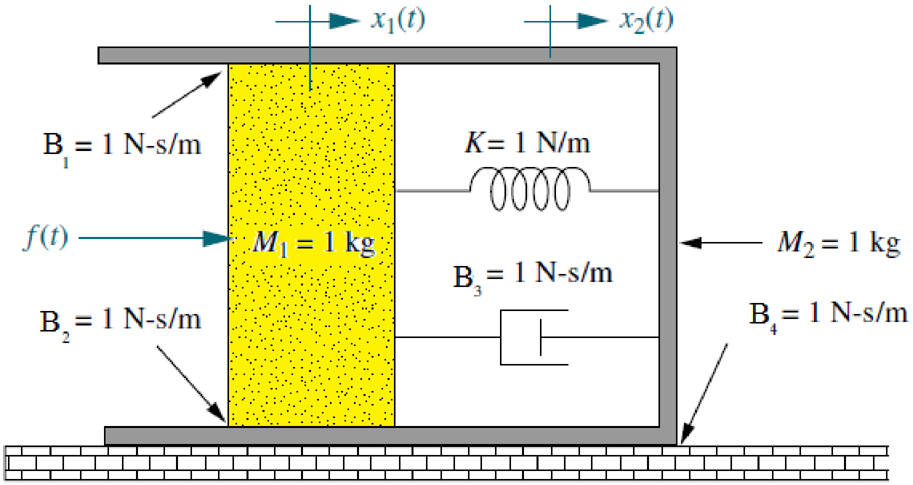




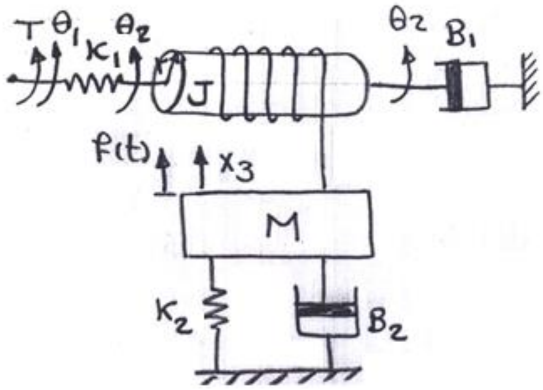






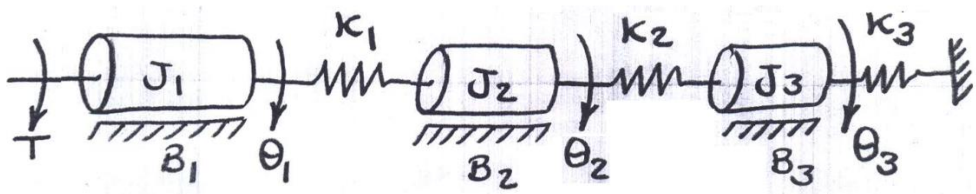


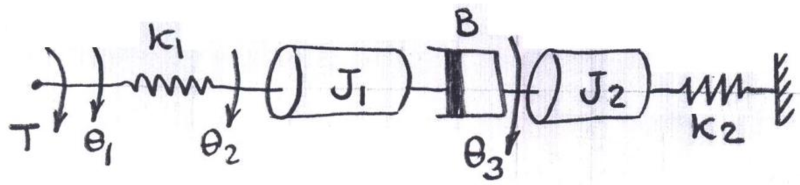
2) Find the D.E's that relates the distance X_3 to θ_1 for the system shown below, then draw the block diagram considering $X_3(s)$ as output. (the radius of the shaft is r).



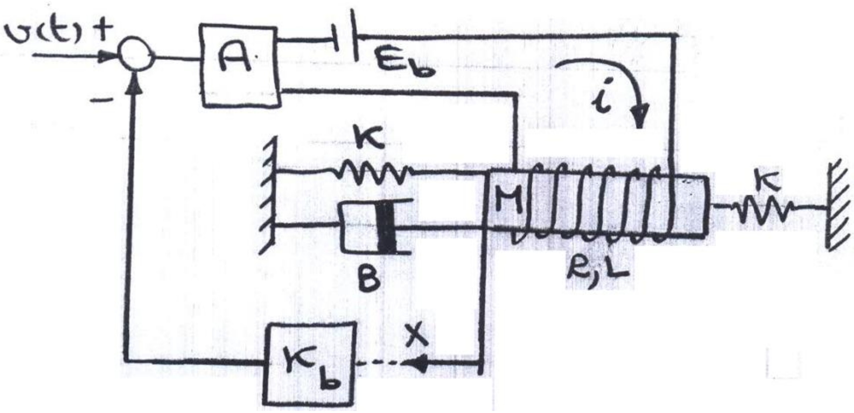
3) For the mechanical systems shown below;

- Draw the mechanical network, then write the system D.E's
- If $\theta_3(s)$ is the system output, draw the block diagram and find $\theta_3(s)/T(s)$

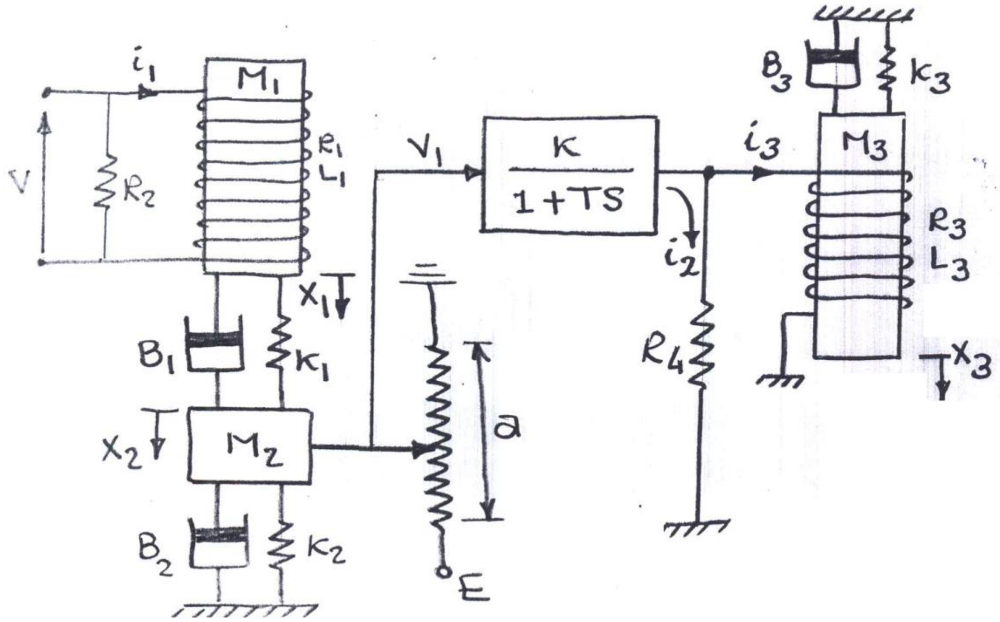




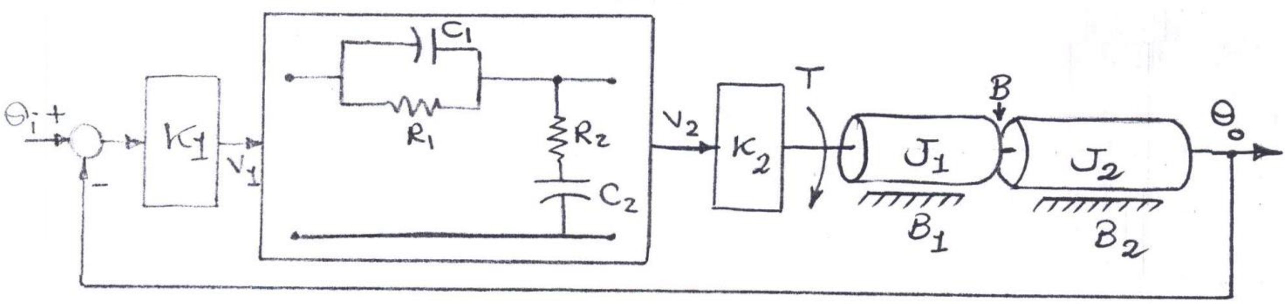
4) For the mechanical system shown below, the solenoid produces a magnetic force $f_c = K_c\, i$. Draw the block diagram then find X(s)/V(s)



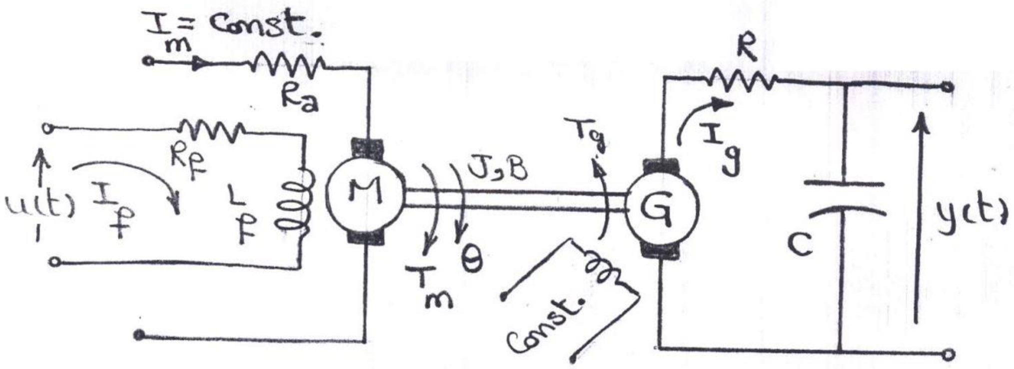
5) The mechanical system shown below is used to measuring the displacement x_2 Write the D.E's describing this system, then draw the block diagram.



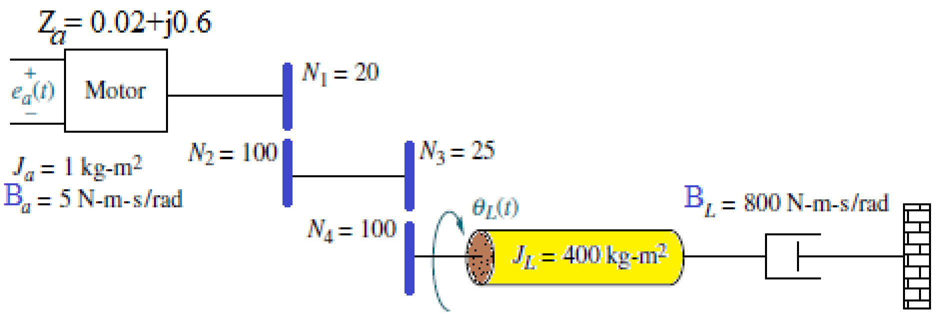
6) For the system shown below, determine the closed loop T.F.



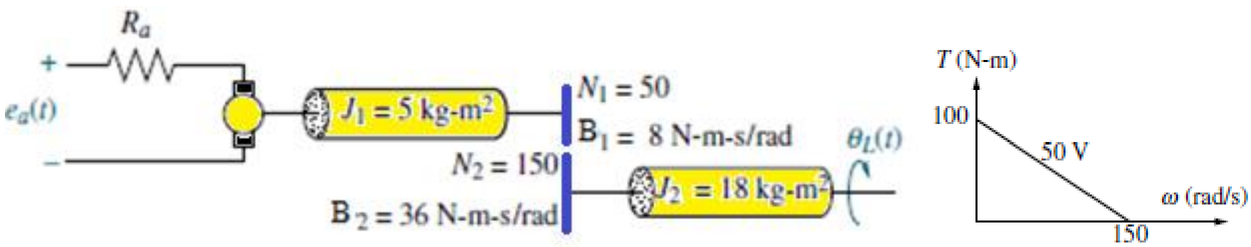
7) For the motor-generator set shown below, the torque constant is K_T for the motor and K_V for the generator. If the generator field current is assumed constant, draw the block diagram then find the T.F. Y(s)/U(s).



8) For the separately-excited DC motor shown below, the torque-speed characteristic is given by $T_m = 200 - 8\omega_m$. Draw the block diagram then find the transfer function $\theta_L(S)/E_a(S)$.



9) For the separately-excited DC motor shown below, the torque-speed characteristic is given. Draw the block diagram then find the transfer function $\theta_L(S)/E_a(S)$.



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