CHAPTER # 7 TIME-DOMAIN ANALYSES

After completing this chapter, the students will be able to:

- Understand the effect of pole and zero on the time response of control system,
- Obtain the time response of first-order systems,
- Write the general equation of the time response of a second-order system in terms of damping ratio and undamped natural frequency,
- Sketch the step response of 2nd order system at different locations of poles,
- Calculate the delay time, rise time, peak time, maximum overshoot & settling time,
- Design a simple PID controller to meet desired time-response specifications.

1. Introduction

It was stated previously in lecture #1 that the first step in analyzing a control system was to derive a mathematical model of the system. Once such a model is obtained, various methods are available for the analysis of system performance.

Typical Test Signals that commonly used as input signals are ***step*** functions, ***ramp*** functions, ***acceleration*** functions, ***impulse*** functions, and ***sinusoidal*** functions. With these test signals, mathematical and experimental analyses of control systems can be carried out easily since the signals are very simple functions of time.

If the inputs to a control system are gradually changing functions of time, then a ramp function of time may be a good test signal. Similarly, if a system is subjected to sudden disturbances, a step function of time may be a good test signal; and for a system subjected to shock inputs, an impulse function may be best. Once a control system is designed on the basis of test signals, the performance of the system in response to actual inputs is generally satisfactory. The use of such test signals enables one to compare the performance of all systems on the same basis.

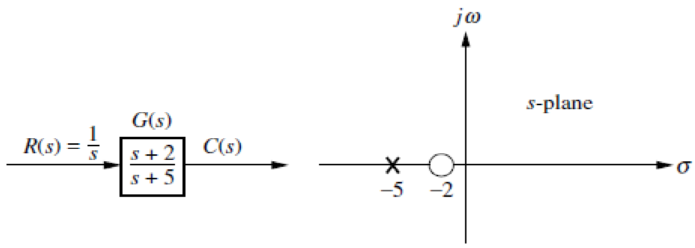
<u>Effect of Pole & Zero on the System Response:</u>

Consider the first-order system that has one real zero at -2 and one real pole at -5, as shown below, therefore the open-loop T.F. is:

$$\frac{C(s)}{R(s)} = \frac{S+2}{S+5}$$

Considering a step input, i.e. R(s) = 1/S;

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

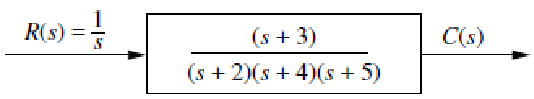


$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

The first part of c(t) is called the <u>forced response</u> (it is independent on time), but the second term is called <u>natural response (it is dependent on time)</u>

Another example: Consider the third-order system with open-loop T.F. is:

$$\frac{C(s)}{R(s)} = \frac{S+3}{(S+2)(S+4)(S+5)}$$



$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}}$$

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}}$$

The *time response* of a control system consists of two parts as shown in Fig. 1;

a) Natural (Transient) response, b) Forced (Steady-state) response.

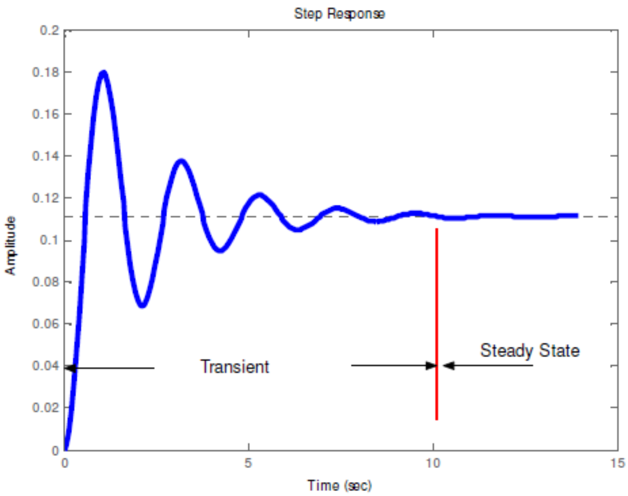


Fig. 1, Time response

Transient response, we mean that system goes from the initial state to the final state.

Steady-state response, we mean the manner in which the system output behaves as t approaches infinity. Thus, the system response $C(t)$ may be written as

$$C(t) = C_{tr}(t) + C_{ss}(t)$$

where $C_{tr}(t)$ is the transient response and $C_{ss}(t)$ is the steady-state response.

The transient response of a practical control system often exhibits damped oscillations before reaching a steady state. If the output of a system at steady state does not exactly agree with the input, the system is said to have steady state error. This error is indicative of the accuracy of the system. In analyzing a control system, we must examine transient-response behavior and steady-state behavior.

2. Transient Response

2.1 First-Order system

Consider the first-order system shown in Fig. 2.

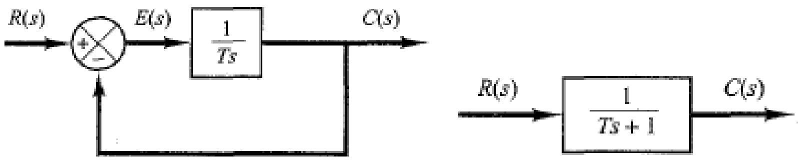


Fig. 2, Block diagram and its simplification

The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{TS + 1}$$

For a unit step input whose Laplace transform is 1/S, the output C(S) is given by

$$C(s) = \frac{1}{TS + 1} \times \frac{1}{S}$$

Using partial fraction,

$$C(s) = \frac{1}{S} - \frac{T}{TS + 1} = \frac{1}{S} - \frac{1}{S + 1/T}$$

Taking the inverse Laplace transform

$$c(t) = 1 - e^{-\frac{t}{\tau}} \qquad for\ t \geq 0, (\tau\ is\ the\ time\ constant = T)$$

The above equation indicates that initially (at $t = 0$) the output $c(t)$ is zero and finally (at $t = \infty$) it becomes unity as shown in Fig. 3.

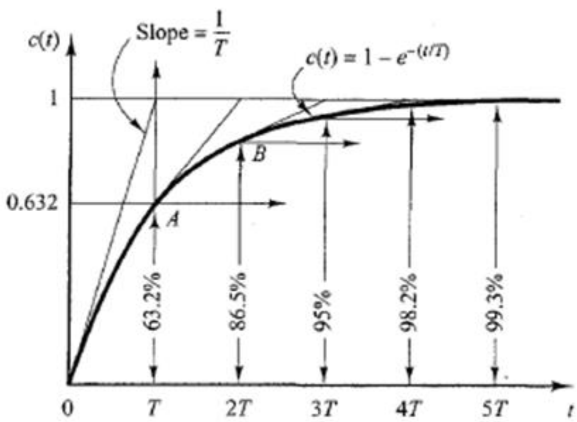


Fig. 3. Time response of a first-order system

One important characteristic of such an exponential response curve $c(t)$ is that at t = T the value of $c(t)$ is 0.632, or the response $c(t)$ has reached 63.2% of its final value. This may be easily seen by substituting t = T in $c(t)$. That is,

$$c(T) = 1 - e^{-1} = 0.632 = 63.2\%$$

By the same way, in two time constants ($t = 2T$), the response reaches 86.5% of the final value. At $t = 3T$, the response reaches 95% of its final value. At $t = 4T$, the system response reaches 98.2% of its final value. Finally, at $t = 5T$, the response reaches 99.3% of the final value. Thus, for $t \geq 4T$, the response remains within 2% of the final value. As seen from the equation of $c(t)$, the steady state value ($c(t) = 1$) is reached mathematically only after an infinite time. In practice, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or four-time constants.

2.1.1 Testing First Order Systems

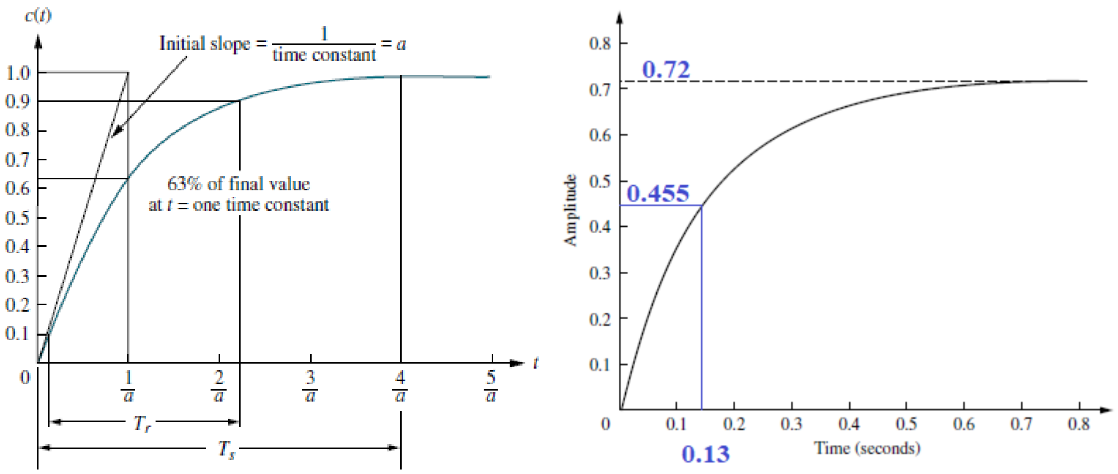


Fig.4, Step response of a first order system

Consider the step response of a first order system shown in Fig. 4 (left) above, the time constant, rise time and settling time are indicated.

By the same way, can you calculate these 3 parameters if the step response is as shown in Fig. 4 (right)?

The T.F. of the first order system is

$$\frac{C(s)}{R(s)} = \frac{K}{1 + TS} = \frac{K/T}{S + 1/T}$$

The final value of this system is 0.72, therefore, 63.2% of this value is 0.632×0.72=0.455, the time corresponding this value is the time constant T=0.13 s.

Since the final value is 0.72 = K → K/T = 0.72÷0.13 = 5.6154

$$\frac{C(s)}{R(s)} = \frac{5.6154}{S + 7.69231}$$

The settling time is after 4T, therefore, $T_s = 4\times0.13=0.52$ s.

2.2 Second-Order Systems

Consider the 2nd order control system shown in Fig. 5, whose T.F. is given as:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{S^2 + 2\zeta\omega_n S + {\omega_n}^2}$$

This form is called the *standard form* of the second-order system, where ζ and ω_n are the damping ratio and undamped natural frequency, respectively.

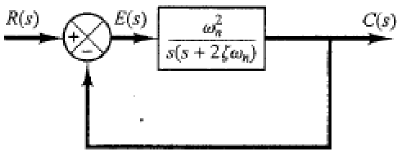


Fig. 5. Standard form of Second-order control system

For a unit-step input ($R(S) = 1/S$), $C(s)$ can be written

$$C(s) = \frac{{\omega_n}^2}{S^2 + 2\zeta\omega_n S + {\omega_n}^2} \times \frac{1}{S}$$

Using partial fraction, Taking inverse Laplace for the output $C(s)$,

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right] - \mathcal{L}^{-1}\left[\frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

$$\therefore c(t) = 1 - e^{-\zeta\omega_n t}\cdot\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\cdot e^{-\zeta\omega_n t}\cdot\sin\omega_d t$$

The frequency ω_d**,** is called the *damped natural frequency*.

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Rewrite the response c(t):

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\cdot\sin\omega_d t\right)$$

Since $\zeta=\cos(\beta)$, therefore $\sin(\beta) = \sqrt{1-\zeta^2}$

$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\beta\cos\omega_d t + \cos\beta\,\sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \beta\right)$$

Where β (in radians) is defined by Fig. 6.

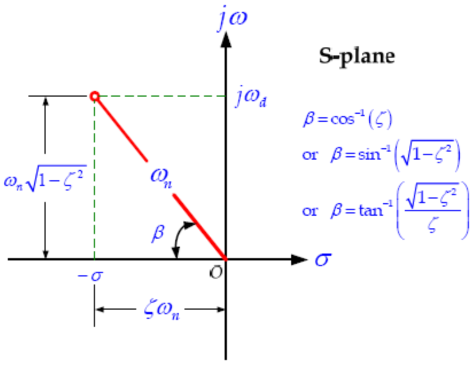


Fig. 6. Definition of angle β

If we plot the output $c(t)$ versus time, such kind of plot is dependent on the two parameters ζ and ω_n. A family of curves at different values of ζ is shown in Fig. 7.

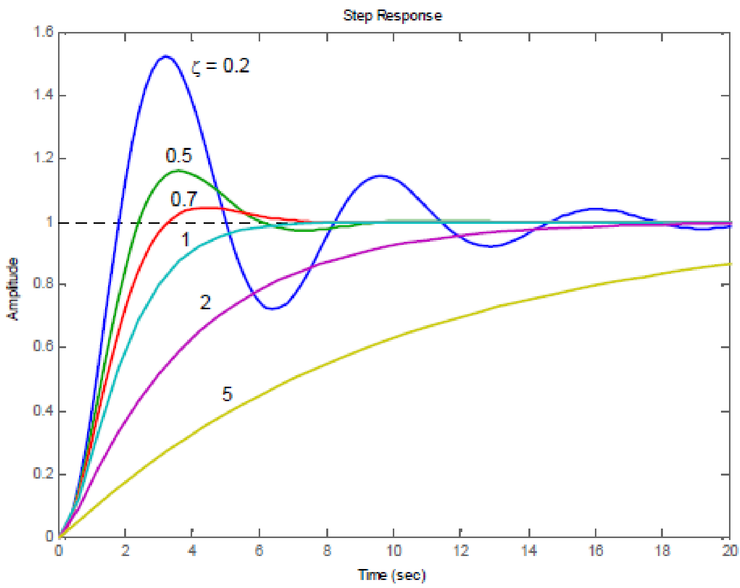


Fig. 7. Transient response of 2nd order system at different ζ.

The characteristic equation of any 2nd order system is given by:

$$S^2 + 2\zeta\omega_n S + {\omega_n}^2 = 0$$

Complete square of the above equation we get;

$$(S + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2 = 0$$

The damping ratio (ζ) is represented by radial line passing through the origin as shown in Fig. 8. For example, a damping ratio of 0.5 requires that the complex-conjugate poles lie on the lines drawn through the origin making angles of ±60° with the negative real axis. (If the real part of a pair of complex-conjugate poles is positive, which means that the system is unstable, the corresponding ζ is negative.) The damping ratio determines the angular location of the poles, while the distance of the pole from the origin is determined by the undamped natural frequency ω_n. The constant ω_n loci are circles.

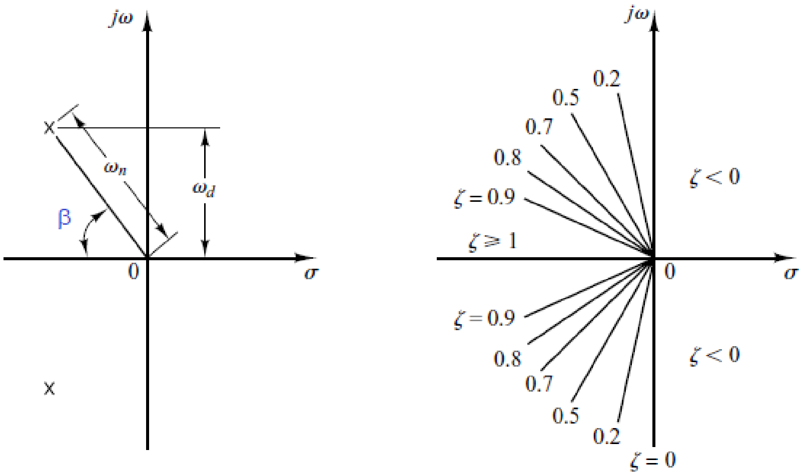
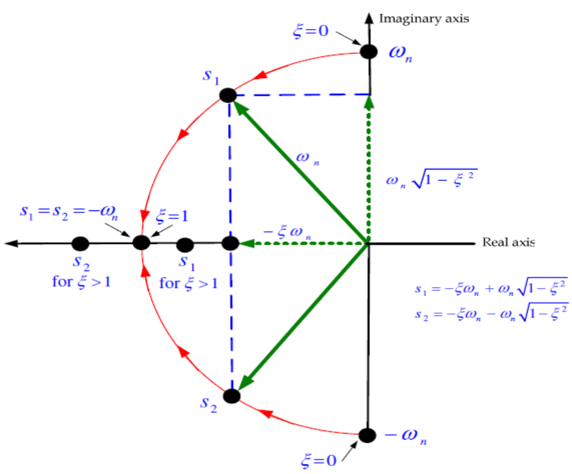


Fig. 8, radial lines represent the damping ratio

As the parameters ζ changes, the location of the system poles S_1 and S_2 are changed. Therefore, the dynamic behavior of the second-order system is also changed. The nature of the roots S_1 and S_2 of the characteristic equation with varying values of damping ratio ζ can be shown in the complex plane as shown in Fig. 9.



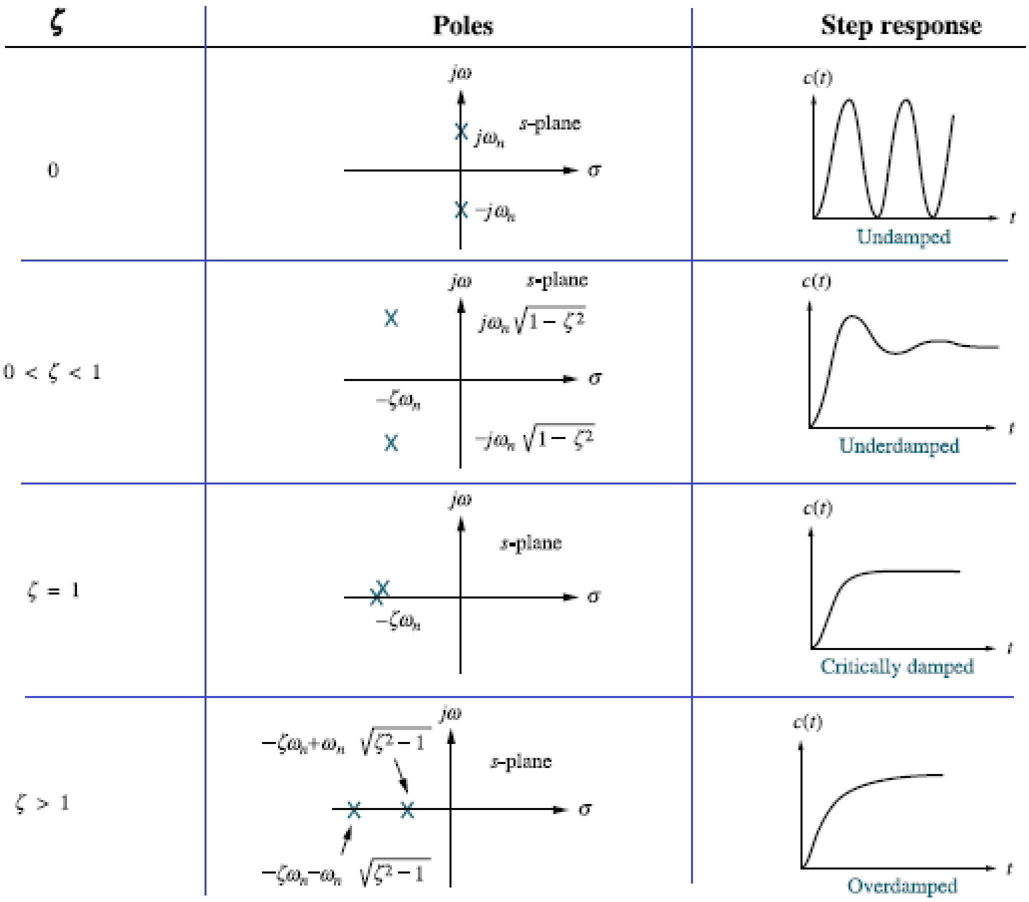


Fig. 9. Closed loop poles and transient response

To summarize our observations. The natural response has four different shapes:

1) Undamped Response: poles are pure imaginary at $\pm j\omega_n$, Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles,

$$C(t) = A\cos(\omega_n t - \varphi)$$

2) Underdamped Response: poles are complex at $\sigma_d \pm j\omega_d$, Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles,

$$C(t) = A\, e^{-\sigma_d t} \cos(\omega_d t - \varphi)$$

3) Critically-damped Response: poles are pure real and equal at $-\sigma_1$, Natural response: One term is an exponential whose time constant is equal to the reciprocal of

the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location,

$$C(t) = A_1\, e^{-\sigma_1 t} + A_2 t\, e^{-\sigma_1 t}$$

4) Overdamped Response: poles are pure real and not equal at $-\sigma_1$, and $-\sigma_2$ Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations,

$$C(t) = A_1\, e^{-\sigma_1 t} + A_2\, e^{-\sigma_2 t}$$

To become familiar with the four responses of second order system, we take a look at numerical examples shown in Fig. 10.

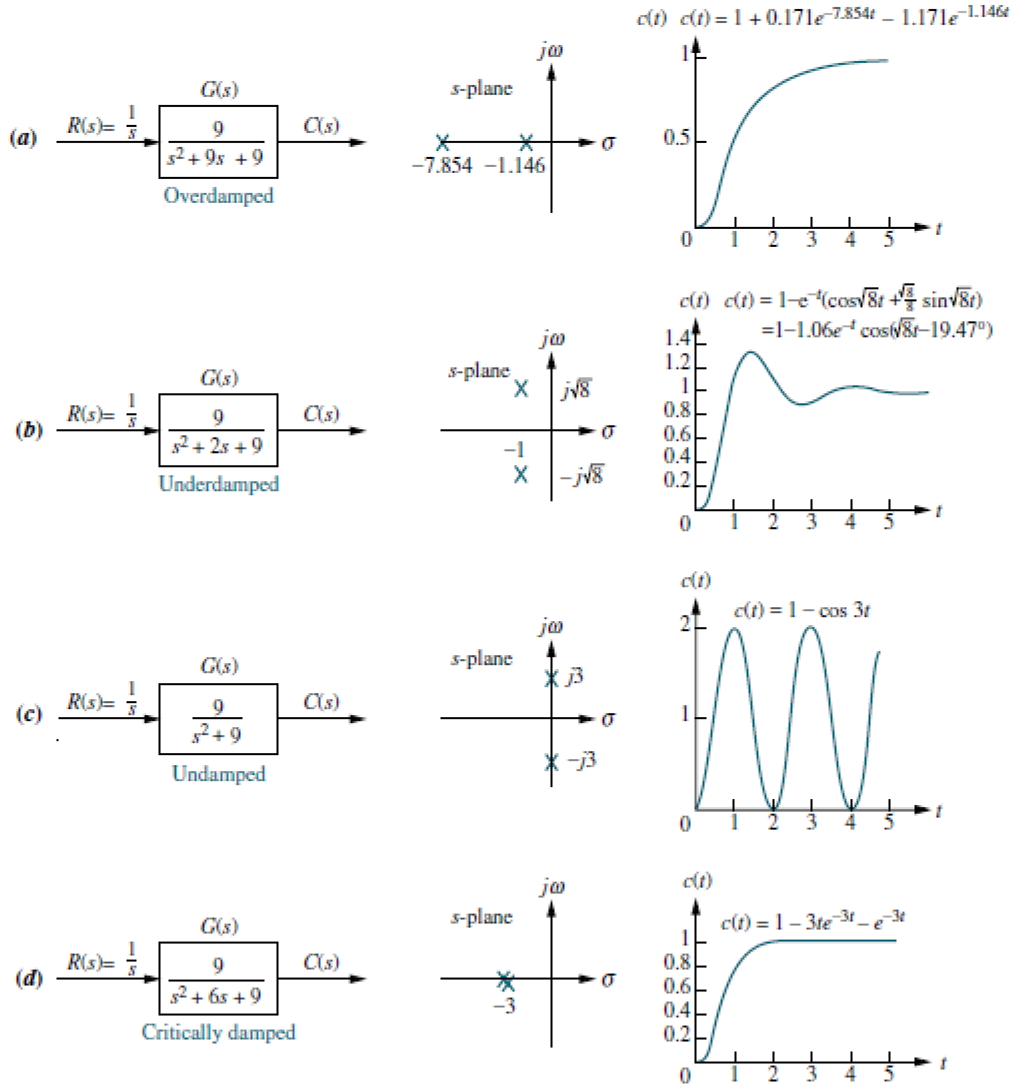


Fig. 10, second-order response at different pole location

2.2.1 Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillations before reaching a steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to name the following terms:

1. Delay Time T_d
2. Rise Time T_r
3. Peak Time T_p
4. Maximum Overshoot $M_p\%$
5. Settling Time T_s

These specifications are shown graphically in Fig. 11.

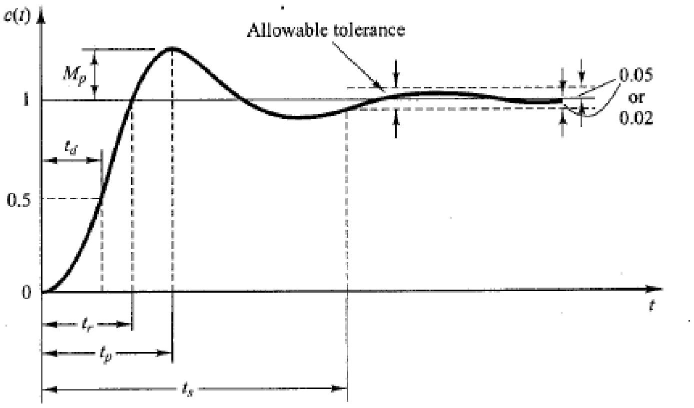


Fig. 11 Transient response specifications

1. Delay Time: The delay time t_d is the time needed for the response to reach half (50%) of its final value. We obtain the delay time t_d by letting $c(t_d) = 0.5$

Delay time can be calculated from this formula;

$$T_d = \frac{1 + 0.7\zeta}{\omega_n}$$

2. Rise Time: The rise time t_r is the time required for the response to rise from 10% to 90%. Or the time required to rise from 0% to 100% of its final value.

We obtain the rise time t_r by letting $c(t_r) = 1$

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right)$$

Since $e^{-\zeta\,\omega_n t_r} \neq 0$, therefore

$$\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r) = 0$$

$$\tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$T_r = \frac{1}{\omega_d}tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \frac{\pi-\beta}{\omega_d}$$

The effect of damping ratio on rise time is shown in Fig. 12.

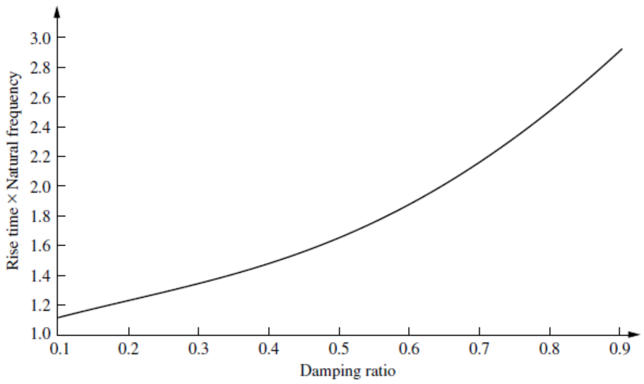


Fig. 12, Rise time versus damping ratio

<u>3. Peak Time:</u> The peak time t_p is the time required for the response to reach the first peak of the overshoot.

We may obtain the peak time by differentiating $c(t)$ with respect to time and letting this derivative equal zero.

$$\frac{dc(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) + e^{-\zeta\omega_n t}\left(\omega_d\sin(\omega_d t) - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos(\omega_d t)\right)$$

The cosine terms in the above equation cancel each other. Therefore, $dc(t)/dt$, evaluated at $t = t_p$, can be simplified to

$$\left.\frac{dc}{dt}\right|_{t=t_p} = \sin(\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

Sin $(\omega_d\, t_p) = 0$, This means $\omega_d\, t_p = 0, \pi, 2\pi, 3\pi, \ldots$

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Each value of *n* yields the time for local maxima or minima. The first peak occurs at peak time (*n*=1) corresponds to the first peak overshoot, $\omega_d\, t_p = \pi$

$$T_p = \frac{\pi}{\omega_d}$$

4. Maximum (percent Overshoot): The maximum percent overshoot M_p is the maximum peak value of the response curve [the curve of *c*(*t*) versus *t*], measured from *c* (∞). If *c* (∞) =1, the maximum percent overshoot is $M_p \times 100\%$. If the final steady-state value *c* (∞) of the response differs from unity, then it is common practice to use the following definition:

$$Max.\%overshoot\ (Mp) = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100\%$$

The maximum overshoot occurs at the peak time. Therefore

$$M_p = C(t_p) - 1 = -e^{-\zeta\omega_n\frac{\pi}{\omega_d}}\left(\cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\pi)\right)$$

$$M_p = e^{-\frac{\pi\,\zeta}{\sqrt{1-\zeta^2}}}$$

The effect of damping ratio on the maximum overshoot is shown in Fig. 13.

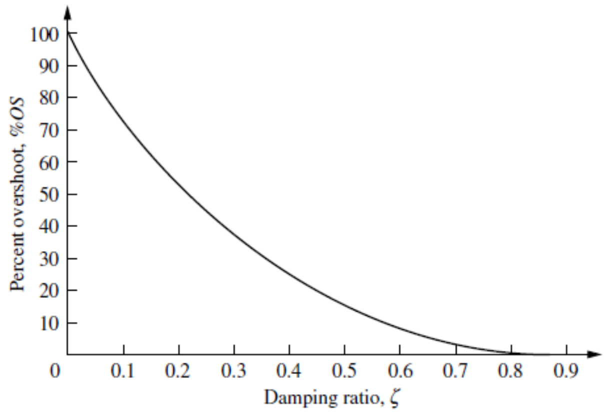


Fig. 13, Percent overshoot versus damping ratio

5. Settling Time: The settling time t_s is the time required for the response curve to reach and stay within ± 2% or ± 5% of the final value. The settling time is the largest time constant of the system.

The settling time corresponding to ± 2% or ± 5% tolerance band may be measured in terms of the time constant $\{T = 1/(\zeta \omega_n)\}$

Based on 2% criteria, it is found that $T_s = 4T$

$$T_s = \frac{4}{\zeta\,\omega_n} \qquad (\pm 2\%\ Criterion)$$

Based on 5% criteria, it is found that $T_s = 3T$

$$T_s = \frac{3}{\zeta\,\omega_n} \qquad (\pm 5\%\ Criterion)$$

Matlab software package can be used to calculate and plot the step response of the second order system. The following m-file can be used.

Effect of moving dominant poles on the response:

1) If the dominant poles are moved vertically with constant real part (Fig. 14)

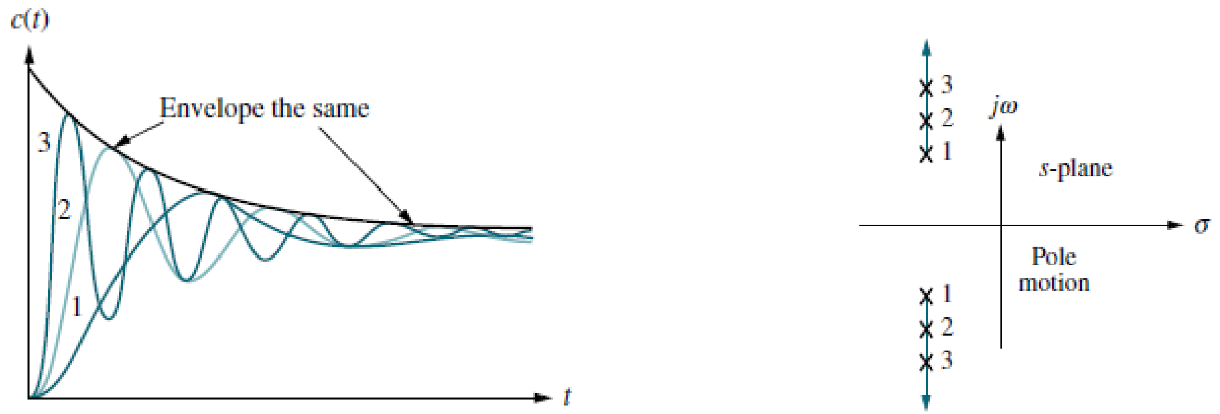


Fig. 14, Moving poles vertically

2) If the dominant poles are moved horizontally with constant imaginary part as shown in Fig. 15.

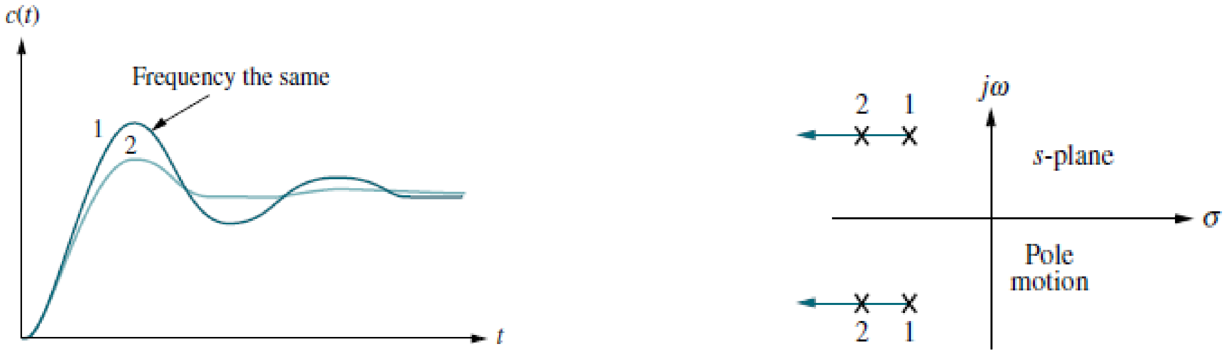


Fig. 15, Moving poles Horizontally

3) If the dominant poles are moved diagonally with constant angle (β) (Fig. 16).

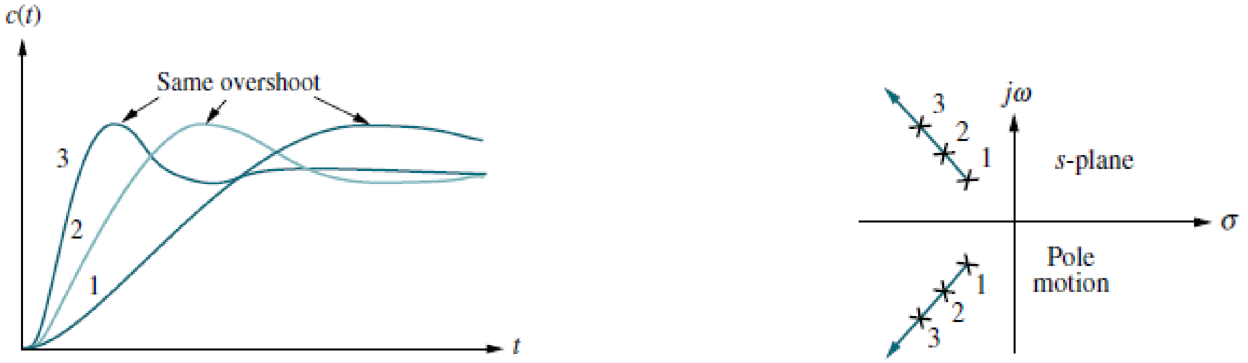


Fig. 16, Moving poles diagonally

4) If the dominant poles are moved on a half circle (ω_n is constant) see Fig. 17.

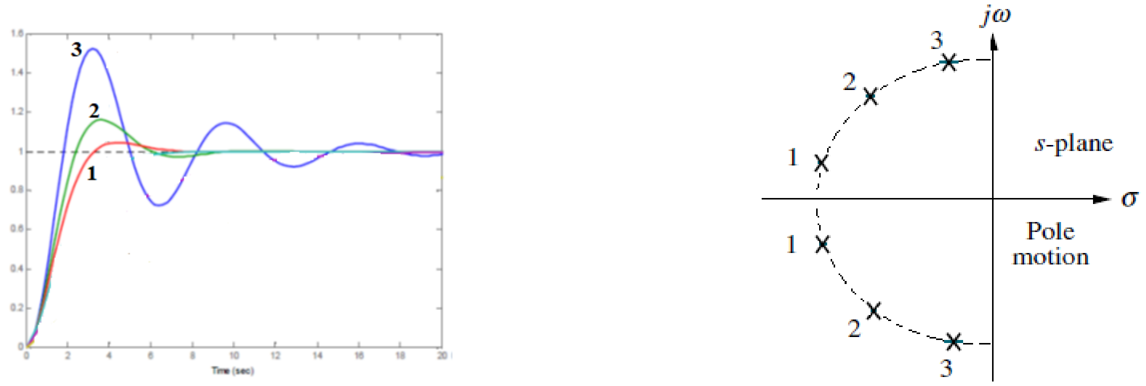
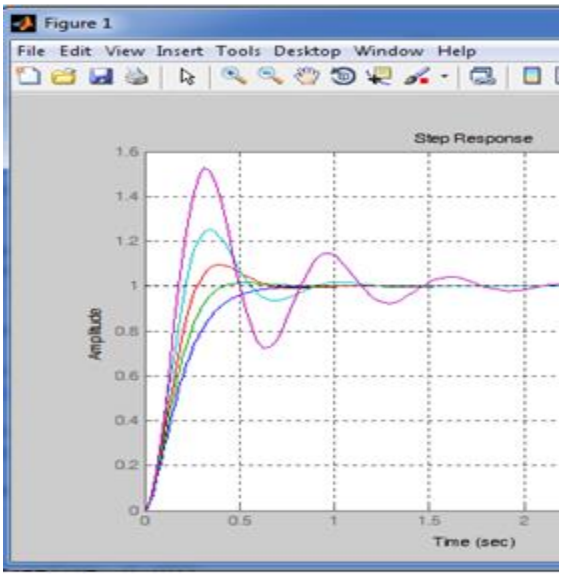
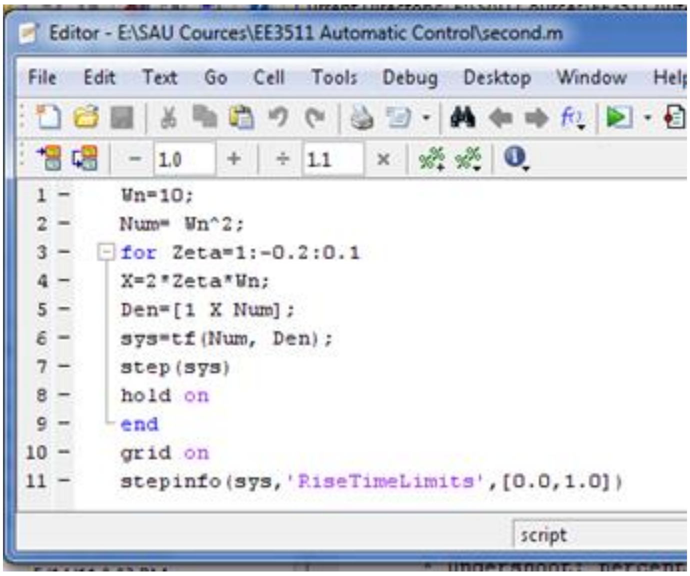


Fig. 17, Moving poles on half a circle

The Matlab code and its run for the last case is given below.





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```
Wn=10;
Num= Wn^2;
for Zeta=1:-0.2:0.1
X=2*Zeta*Wn;
Den=[1 X Num];
sys=tf(Num, Den);
step(sys)
hold on
end
grid on
stepinfo(sys,'RiseTimeLimits',[0.0,1.0])
```

In the previous Matlab code we consider some Matlab functions such as tf and step. What is tf and step? and how can we use them?

"tf" Specifies a SISO transfer function for model h(s) = n(s)/d(s) where n(s) and d(s) represent the numerator and denominator polynomials, respectively.

```
>> h = tf(num, den)
```

<u>What are num & den?</u>
row vectors listing the coefficients of the polynomials $n(s)$ and $d(s)$ ordered in descending powers of s

tf Example

$$T(s) = \frac{2s-3}{s+1} \equiv \texttt{h=tf([2 -3], [1 1])}$$

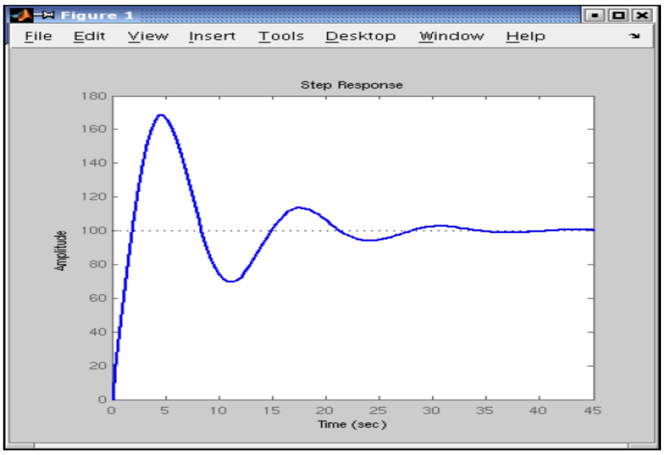
$$T(s) = \frac{2s+1}{4s^2+s+1} \equiv \texttt{h=tf([2 1], [4 1 1])}$$

Step Response

draw the step response of the T.F

$$t.f. = \frac{100(2S+1)}{4S^2+S+1}$$

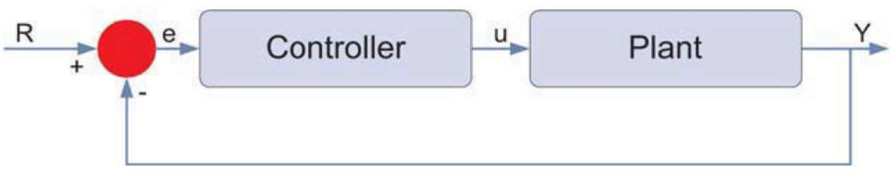
```
system=tf([2 1],[4 1 1])
t=0:0.1:50;
step(100*system)
axis([0 45 0 180])
```



Steady-State Error

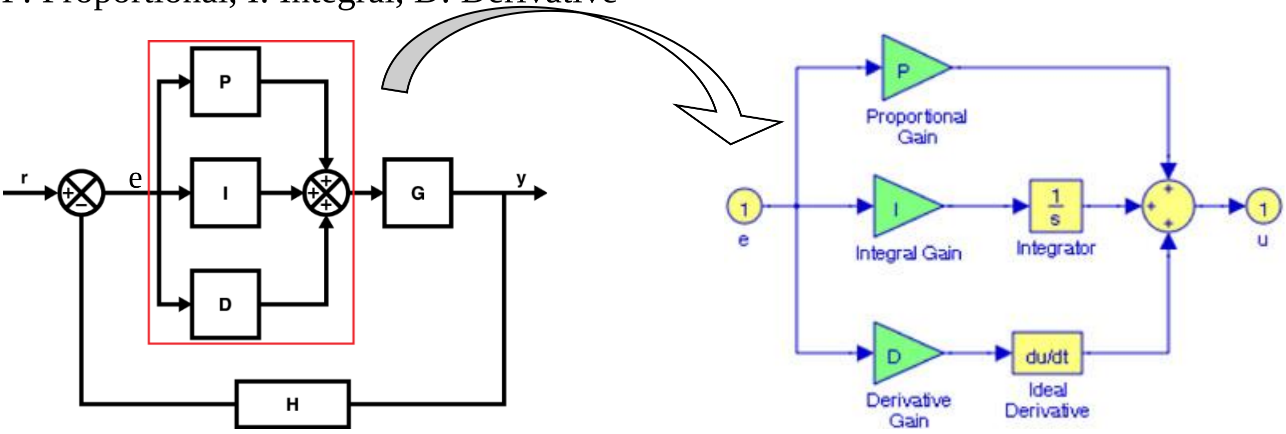
The difference between the input and output of a system in the limit as time goes to infinity, and it will be discussed in more details in next chapter.

<u>Feedback PID controller – How does it work?</u>



As shown in the feedback control system given above, the type of controller used is PID controller. The PID terms are stand for:

P: Proportional, I: Integral, D: Derivative



- e – represents the tracking error
- e – difference between desired input (R) an actual output (Y)
- e – is sent to controller which computes:
  - derivative of e
  - integral of e
- u – controller output is equal to:
  - K_p (proportional gain) times the magnitude of the error +
  - K_i (integral gain) times the integral of the error +
  - K_d (derivative gain) times the derivative of the error

Controller's Output

$$u = K_p e + K_i \int e dt + K_d \frac{de}{dt}$$

Controller's Transfer Function

$$K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s}$$

Characteristics of PID Controllers

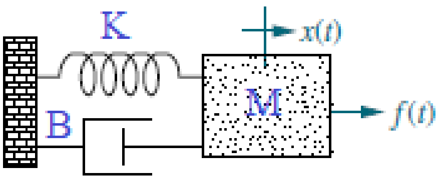
- Proportional Controller K_p
  - reduces the rise time
  - reduces but never eliminates steady-state error
- Integral Controller K_i
  - eliminates steady-state error
  - worsens transient response

- Derivative Controller K_d
  - increases the stability of the system
  - reduces overshoot
  - improves transient response

Type	Rise time	Overshoot	Settling time	S-S Error
K_p	decrease	increase	small change	decrease
K_i	decrease	increase	increase	eliminate
K_d	small change	decrease	decrease	small change

These correlations may not be exactly accurate, because *Kp*, *Ki*, and *Kd* are dependent on each other. In fact, changing one of these variables can change the effect of the other two.

Example Problem



Consider the Mass (M), spring (K), and damper (B) problem given above;

Modelling Equation

$$m\ddot{x} + b\dot{x} + kx = F$$

Neglecting initial values and using Laplace,

$$ms^2X(s) + bsX(s) + kX(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

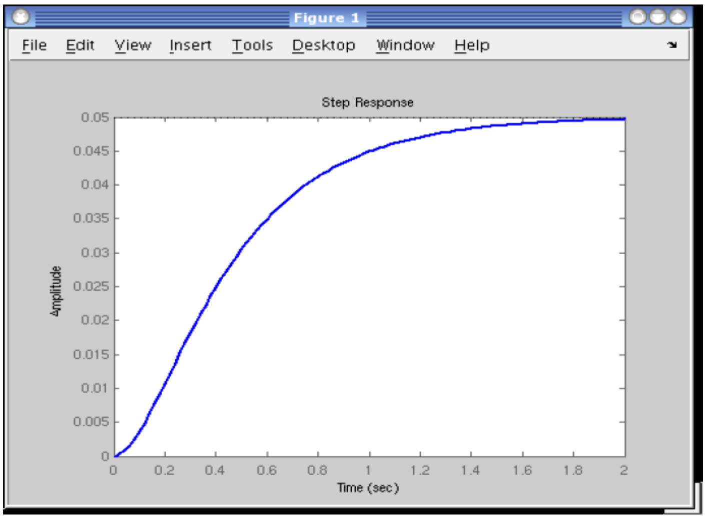
MATLAB System Response

Assumptions

Let: $m = 1[kg]$, $b = 10[Ns/m]$, $k = 20[N/m]$

MATLAB code

```
%{Set up variables%}
m=1; b=10; k=20;
%{Calculate response%}
num=1;
den=[m, b, k];
plant=tf(num,den);
step(plant)
```



From the system response shown above, the Mass-spring and damper system, is suffering from the following problems:

- The steady-state error is equal to 0.95
- The rise time is about 1 second
- The settling time is about 1.5 seconds
- The PID controller should influence (reduce) all those parameters

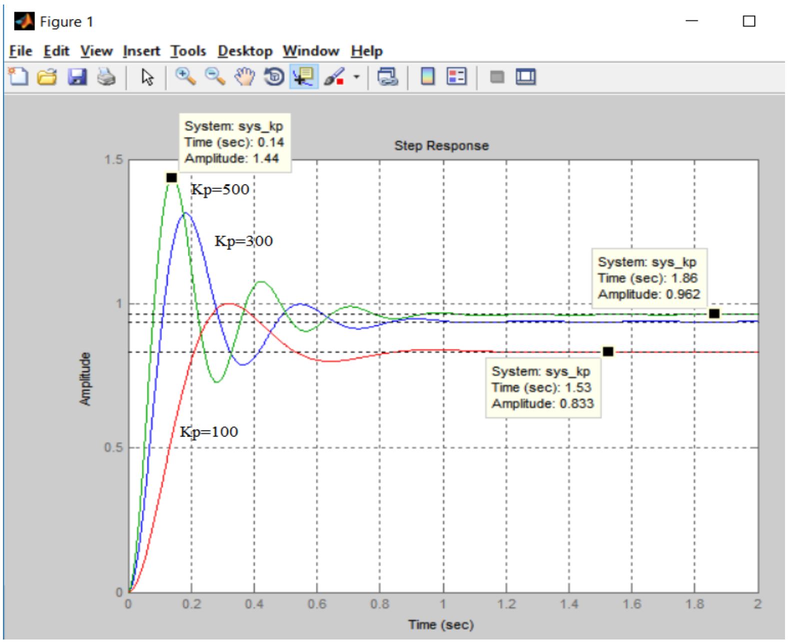
First Trial to solve the system problems is by using Proportional Controller;

P Transfer Function

$$\frac{X(s)}{F(s)} = \frac{K_p}{s^2 + bs + (k + K_p)}$$

MATLAB code

```
M=1;      % Mass of the plant
B=10;     % Friction
K=20;     % Spring
num=1;
den=[M B K];
plant=tf(num,den);
Kp=300;     % set up the proportional gain
sys_kp=feedback(Kp*plant,1);  % canonical TF
t=(0:0.001:2);    % set the time range
hold on
step(sys_kp,t)    % Plot result
grid on
```



Comments on the above figure:

By increasing k_p, rise time is improved (T_r=0.1) and steady-state error is improved (E_{ss}=0.038) but the system overshoot is deteriorated (M_p~1.1). Settling time (T_s=1.2)

Second trial to solve the system problems raised from using Kp only is by using

Proportional Integral Controller

PI Transfer Function

$$\frac{U(s)}{E(s)} = \frac{K_p S + K_i}{S}$$

$$\frac{X(s)}{F(s)} = \frac{K_p s + K_i}{s^3 + bs^2 + (k + K_p)s + K_i}$$

MATLAB code

Editor - G:\SAU Cources\EE3511 Automatic Control\exams\1437_1438\First Term\EE3511pid.m

```
M=1;      % Mass of the plant
B=10;     % Friction
K=20;     % Spring
num=1;
den=[M B K];
plant=tf(num,den);
Kp=50;     % set up the proportional gain
Ki=300;      % set up the integral gain
PIcont=tf([Kp Ki],[1 0]); %TF of PI controller
sys_PI=feedback(PIcont*plant,1);  % canonical TF
t=(0:0.001:2);     % set the time range
hold on
step(sys_PI,t)    % Plot result
grid on
```

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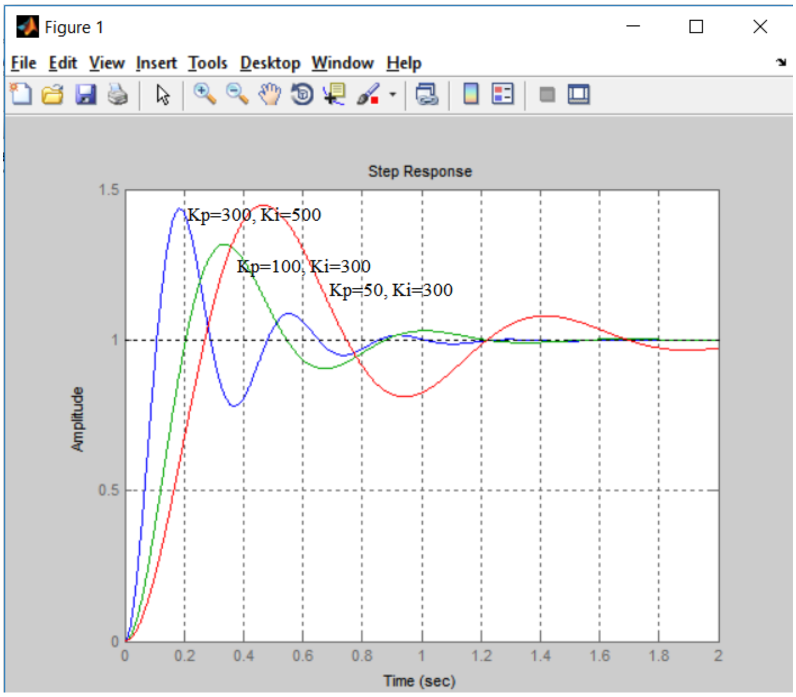


Comments on the above figure:

It is important to note that: Eliminated steady-state error, decreased over-shoot But rise and settling times (T_r & T_s) are deteriorated

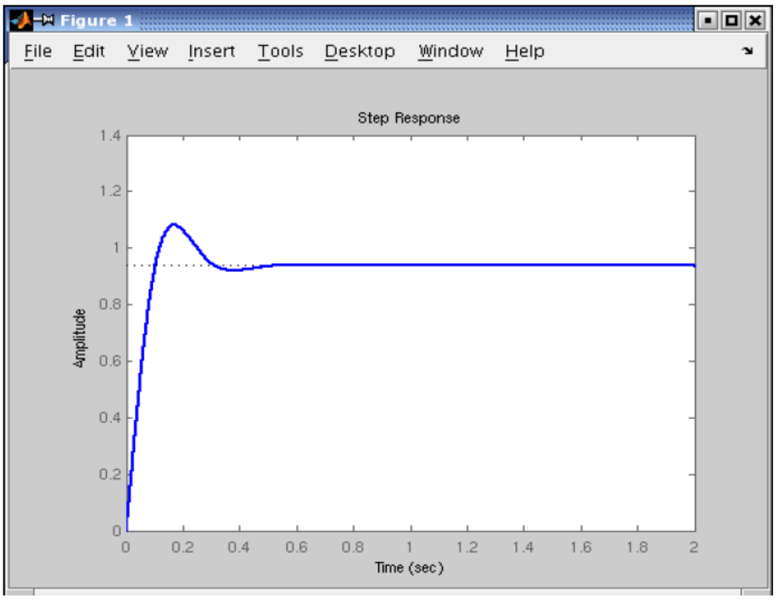
Third Trial to solve the system problems is by using Proportional-Derivative Controller;

PD Transfer Function

$$\frac{X(s)}{F(s)} = \frac{K_d s + K_p}{s^2 + (b + K_d)s + (k + K_p)}$$

MATLAB code

```
num=1;
den=[m,  b,  k];
plant=tf(num,den);
%{Set up proportional and derivative gain}
Kp=300;   Kd=10;
%{Calculate controller}
contr=tf([Kd,  Kp],1);
sys_ctl=feedback(contr*plant,1);
%{Plot results%}
t=0:0.01:2;
step(sys_ctl,t)
```



Comments on the above figure:

Rise time and steady-state error are not affected. But the system overshoot is improved (M_p~1.05) and settling time is improved (T_s~0.5)

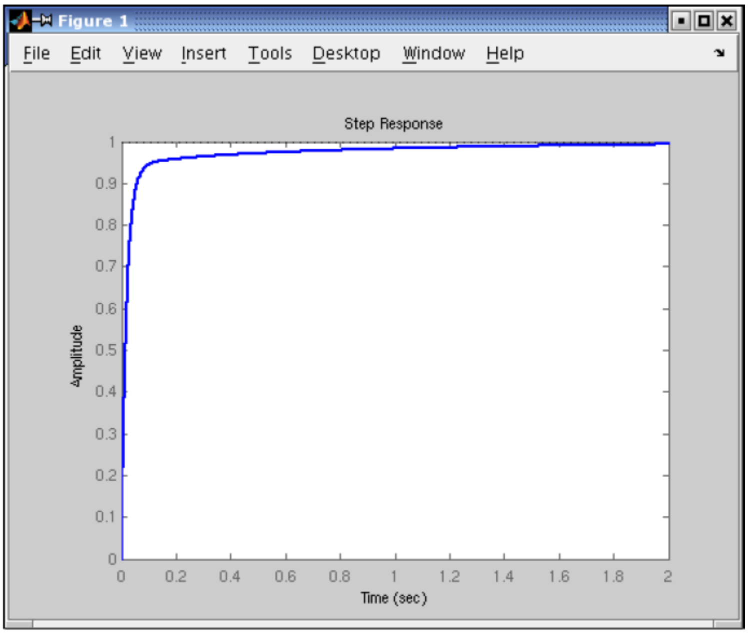
Fourth Trial to solve the system problems is by using Proportional-Integral-Derivative (PID) Controller;

PID Transfer Function

$$\frac{X(s)}{F(s)} = \frac{K_d s^2 + K_p s + K_i}{s^3 + (b + K_d)s^2 + (k + K_p)s + K_i}$$

MATLAB code

```
num=1;
den=[m, b, k];
plant=tf(num,den);
%{Set up proportional and integral gain}
Kp=350; Ki=300; Kd=50;
%{Calculate controller}
contr=tf([Kd, Kp, Ki],[1, 0]);
sys_ctl=feedback(contr*plant, 1);
%{Plot results%}
t=0:0.01:2;
step(sys_ctl, t)
```

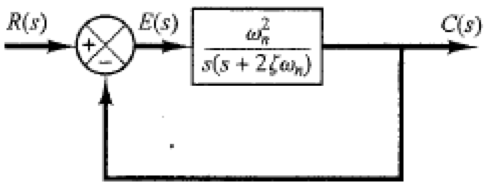


Comments on the above figure:

It is important to note that: Eliminated steady-state error, decreased over-shoot Also rise and settling times (T_r & T_s) are improved

Example #1

Consider the system shown below, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.



$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$$

$$\sigma = \zeta\omega_n = 3.$$

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time t_r is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling time t_s: For the 2% criterion, the settling time is

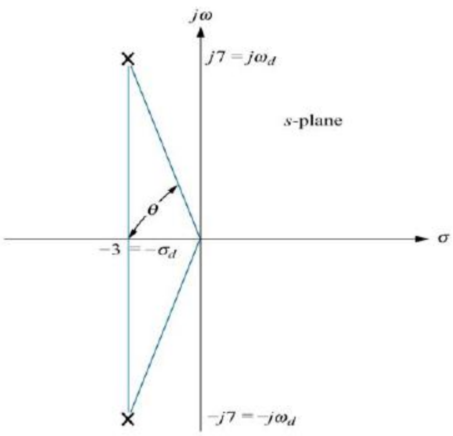
$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

Example #2

Consider the control system whose closed loop poles are given below.



Find $\zeta, \omega_n, T_p, \%OS$, and T_s.

Solution The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second}$$

The percent overshoot is

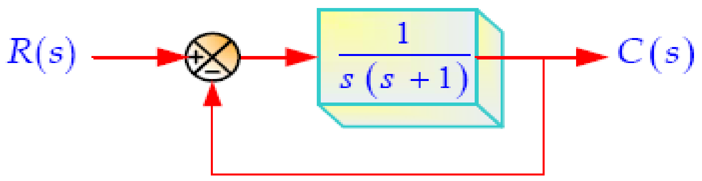
$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26.018\%$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds}$$

Example #3

Determine the values of T_d, T_r, T_p *and* T_s for the control system shown below.



The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s(s+1)}}{1+\frac{1}{s(s+1)}} = \frac{1}{s^2+s+1}$$

Notice that $\omega_n = 1$ rad/s and $\zeta = 0.5$ for this system. So $\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{1-0.5^2} = 0.866$

The rise time is given by

$$T_r = \frac{\pi - \beta}{\omega_d}$$

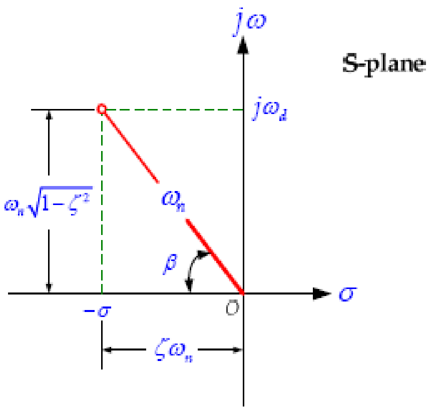
So, we must calculate the angle β first based on Fig. below, as follows:

$\beta = \sin^{-1}(\omega_d/\omega_n) = \sin^{-1}(0.866/1) = 1.05$ rad

or

$\beta = \cos^{-1}(\zeta\omega_n/\omega_n) = \cos^{-1}(\zeta) = \cos^{-1}(0.5)$

$= 1.05$ rad



$$T_r = \frac{\pi - 1.05}{0.866} = 2.41 \text{ s}$$

Peak Time. $T_p = \frac{\pi}{\omega_d} = \frac{\pi}{0.866} = 3.63$ s

Delay Time.

$$T_d = \frac{1+0.7\zeta}{\omega_n} = \frac{1+0.7(0.5)}{1} = 1.35 \text{ s}$$

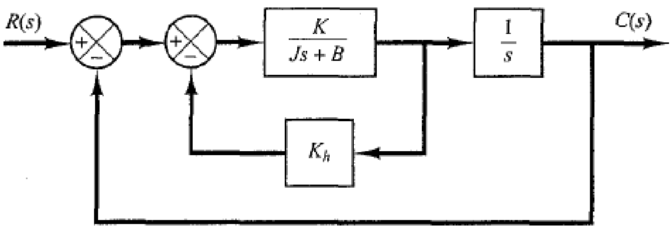
Maximum Overshoot : $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\pi\times 0.5/\sqrt{1-0.5^2}} = e^{-1.81} = 0.163 = 16.3\%$

Settling time: $T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5\times 1} = 8$ s

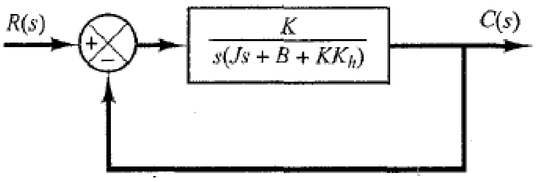
Example #4

For the system shown below, determine the values of gain K and velocity feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec.

With these values of K and K_h, obtain the rise time and settling time. Assume that J = 1 kg-m^2 and B = 1 N-m/rad/sec.



The simplified block diagram of the system is:



The overall T.F. is given by:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

By comparing, we find that

$$\omega_n = \sqrt{K/J} \qquad \zeta = \frac{B + KK_h}{2\sqrt{KJ}}$$

Since $M_p = 0.2$;

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

Since Peak time = 1, then

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since ζ is 0.456, ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53$$

Since the natural frequency ω_n is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then k_h can be determined as:

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

Therefore the rise time (t_r) can be calculated as:

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}1.95 = 1.10$$

Therefore, T_r=0.65 sec.

Settling time t_s: For the 2% criterion,

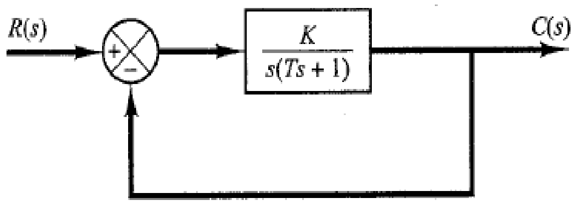
$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

For the 5% criterion,

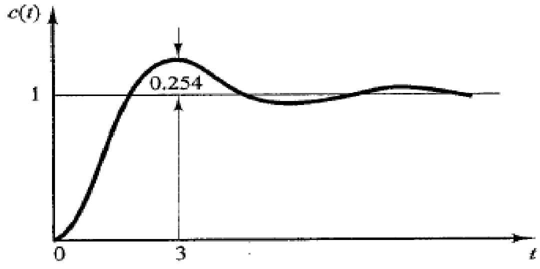
$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$

Example #5

When the system shown in Fig. (a) is subjected to a unit-step input, the system output responds as shown in Fig. (b). Determine the values of K and T from the response curve.



(a)



(b)

From the time response curve we can obtain that:

$M_p = 0.254 \Longrightarrow \zeta = 0.4$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = 3$$

It follows that

$$\omega_n = 1.14$$

From the block diagram we have

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

from which

$$\omega_n = \sqrt{\frac{K}{T}}, \qquad 2\zeta\omega_n = \frac{1}{T}$$

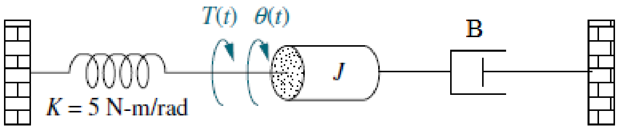
Therefore, the values of T and K are determined as

$$T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} = 1.09$$

$$K = \omega_n^2 T = 1.14^2 \times 1.09 = 1.42$$

Example #6

Given the system shown below, find inertia (J) and friction (B) to yield 20% overshoot and a settling time (based on ±2% criterion) of 2 seconds for a unit step input of torque T(t).



The DE represent that system is

$$T(s) = \theta(s)\{JS^2 + BS + K\}$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{JS^2 + BS + K}$$

The system characteristic eqn. is given by

$$JS^2 + BS + K = 0$$

$$S^2 + \frac{B}{J}S + \frac{K}{J} = 0$$

The standard form of 2nd order system is

$$S^2 + 2\zeta\omega_n S + \omega_n^2 = 0$$

By comparing coefficients:

$$\omega_n = \sqrt{\frac{K}{J}} \qquad (1)$$

$$2\zeta\omega_n = \frac{B}{J} \qquad (2)$$

But it is given that $T_s = 2$ for ±2% criterion,

$$T_s = 2 = \frac{4}{\zeta\omega_n} \rightarrow \zeta\omega_n = 2$$

From (2), B/J = 4

It is given that $M_p = 0.2 \rightarrow \zeta = 0.456$

From (1), $\zeta = 2\sqrt{J/K}$

$$\sqrt{\frac{J}{K}} = \frac{0.456}{2} \rightarrow J = \left(\frac{0.456}{2}\right)^2 \times K = 0.26\ kg.m^2$$

$$\frac{B}{J} = 4 \ \rightarrow B = J \times 4 = 1.04\ N.m.s/rad$$

Example #7

The T.F. of a closed-loop, unity feedback control system is

$$\frac{C(S)}{R(S)} = \frac{K}{S^2 + \ 2S + K}$$

If the system gain (K) is set at three different values of 10, 36 and 100

- Calculate the rise time, maximum overshoot, and settling time at each value of K,
- At which value of K the system response is superior.

This is a good example for proportional controllers (P-Controllers)

The general form of the second-order system is

$$\frac{C(S)}{R(S)} = \frac{\omega_n{}^2}{S^2 + \ 2\xi\omega_n S + \omega_n{}^2}$$

By Comparing,

$\omega_n = \sqrt{K}$

$\xi = 1 / \sqrt{K}$

$B = \cos^{-1} (1 / \sqrt{K})$

$\omega_d = \omega_n \sqrt{1 - \xi^2} = \sqrt{K(1-1/k)} = \sqrt{(K-1)}$

Based on 2% criteria, it is found that $T_s = 4T$,

At K = 10

$\omega_n = \sqrt{10} = 3.1623$

$\xi = 1 / \sqrt{10} = 0.31623$

$B = \cos^{-1} (1 / \sqrt{10}) = 71.56505^\circ = 1.24904577$ rad

$\omega_d = \sqrt{K-1} = 3.0$

$$\text{Rise Time (Tr)} = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.24904577}{3} = 0.63085\ sec$$

$$Maximum\ overshoot\ Mp = \ e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.35085 = 35.085\%$$

$$T_s = \frac{4}{\xi\ \omega_n} = \frac{4}{0.31623 \times 3.1623} = 4$$

At K = 36

$\omega_n = \sqrt{36} = 6$

$\xi = 1 / 6 = 0.16667$

B = $\cos^{-1}$ (1 / 6) = 80.40593177 ° = 1.4334825 rad
$\omega_d = \sqrt{35} = 5.9160798$

$$\text{Rise Time (Tr)} = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.4334825}{\sqrt{35}} = 0.2938\ sec$$

$$Maximum\ overshoot\ Mp = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.588 = 58.8\%$$

$$T_s = \frac{4}{\xi\ \omega_n} = \frac{4}{0.16667 \times 6} = 4\ sec$$

At K = 100
$\omega_n = \sqrt{100} = 10$
$\xi = 1 / 10 = 0.1$
B = $\cos^{-1}$ (0.1) = 84.261 ° = 1.47063 rad
$\omega_d = \sqrt{99} = 9.94987$

$$\text{Rise Time (Tr)} = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.47063}{\sqrt{99}} = 0.167938\ sec$$

$$Maximum\ overshoot\ Mp = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.72925 = 72.925\%$$

$$T_s = \frac{4}{\xi\ \omega_n} = \frac{4}{0.1 \times 10} = 4\ sec$$

	Rise Time	Maximum Overshoot	Settling Time
10	0.63085 sec	35.085%	4 sec
36	0.2938 sec	58.8%	4 sec
100	0.167938 sec	72.925%	4 sec

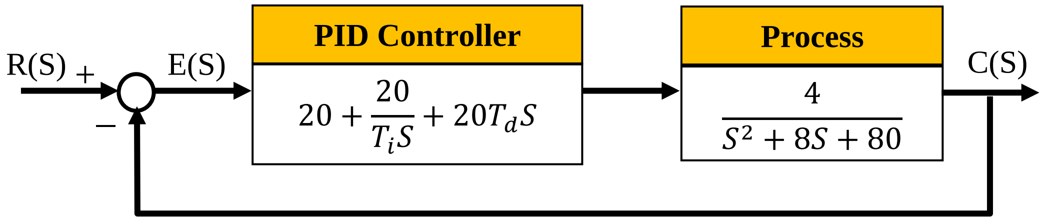
Based on information given in the table, by increasing the system gain from 10 to 100, the rise time and steady-state error are decreased (improved) which is V.Good. On the other hand, the Maximum overshoot is increased (deteriorated).

Example #8

A 3-term (PID) controller is used to control a process with unity feedback as shown in Fig. 3, where T_i and T_d are the integral and derivative time constant, respectively. For unit step input,

a) If $T_d = 3.5$, and the integral term is ignored, calculate the steady-state error,
b) If $T_i = 2.0$, and T_d as given in (a), calculate the steady-state error,
c) Which steady-state error obtained from (a) and (b) is better. Why?

d) If both derivative and integral terms are ignored, calculate the damping ratio, maximum overshoot, rise time, peak time and settling time, then draw a free-hand sketch for the system output c(t).



H(S) = 1,

$$G(S) = \frac{80\left(1 + \frac{1}{T_i S} + T_d S\right)}{S^2 + 8S + 80}$$

Since unit step input, we calculate the position error coefficient K_p

a) T_i is set to ∞ to ignore the integral term

$T_d = 3.5$

$$G(S) = \frac{80(1 + 3.5\, S)}{S^2 + 8S + 80}$$

$$K_p = \lim_{S \to 0} G(S) = \frac{80}{80} = 1$$

$$E_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + 1} = 0.5$$

b) $T_i = 2.0$ and $T_d = 3.5$

$$G(S) = \frac{80\left(1 + \frac{1}{2S} + 3.5\, S\right)}{S^2 + 8S + 80}$$

$$K_p = \lim_{S \to 0} G(S) = \frac{\infty}{80} = \infty$$

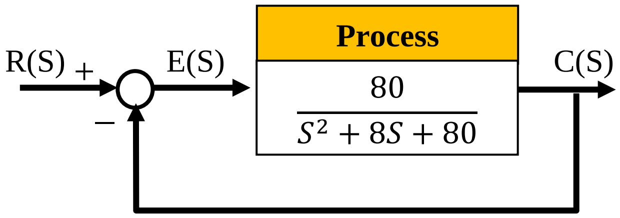
$$E_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0$$

c) the steady-state error in case (b) is better than that of (a) because the integral term is employed, therefore the system type is increased by one, so that the error is reduced to 0.

d) T_i is set to ∞ to ignore the integral term

T_d is set to 0 to ignore the derivative term

The overall system is shown in the figure below



$$\frac{C(S)}{R(S)} = \frac{80}{S^2 + 8S + 160}$$

The system characteristic equation is

$$S^2 + 8S + 160 = 0$$

The standard form of second order system characteristic equation is

$$S^2 + 2\zeta\omega_n\, S + {\omega_n}^2 = 0$$

By comparing the coefficients

$\omega_n = \sqrt{160} = 12.649$ rad/sec

$2\xi\, \omega_n = 8 \rightarrow \xi = 0.3162$

Maximum overshoot = $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$ = 0.35096 = 35.096 %

$\beta = \cos^{-1} 0.3162 = 71.5667° = 1.2491$ rad

Rise Time T_r

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.2491}{12.649\sqrt{1 - 0.3162^2}} = 0.1577\ sec.$$

Peak Time T_p

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{12.649\sqrt{1 - 0.3162^2}} = 0.2618\ sec.$$

Settling Time T_s

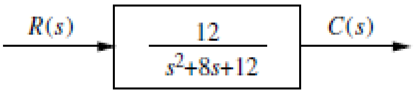
$$T_s = \frac{3}{\zeta\,\omega_n} = \frac{3}{0.3162 \times 12.649} = 0.75\ sec.\ (Based\ on\ \pm 5\%\ tolerance)$$

$$T_s = \frac{4}{\zeta\,\omega_n} = \frac{4}{0.3162 \times 12.649} = 1.0\ sec.\ (Based\ on\ \pm 2\%\ tolerance)$$

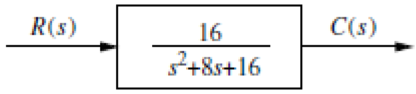
Sheet 6 (Transient Response)

Problem #1

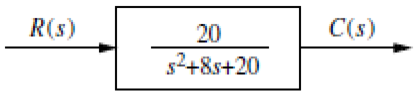
For each of the systems shown below, find the value of ζ and report the kind of response expected.



(a)



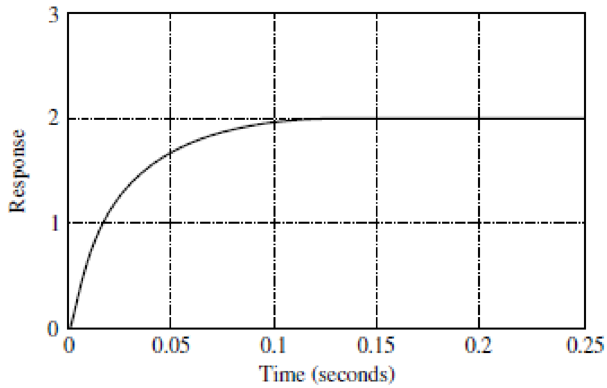
(b)

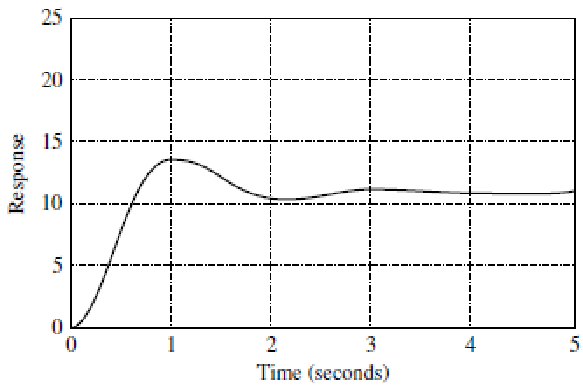


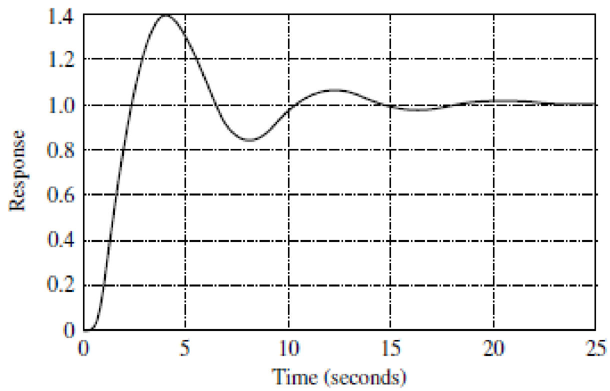
(c)

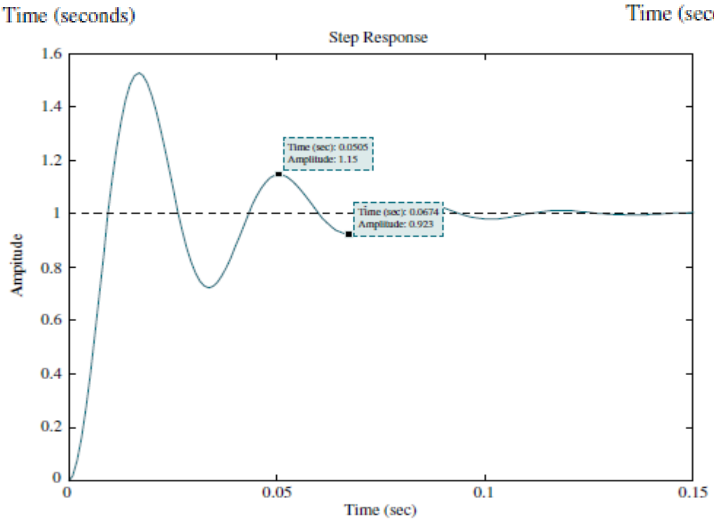
Problem #2

For each of the unit step responses shown below, find the transfer function of the system.





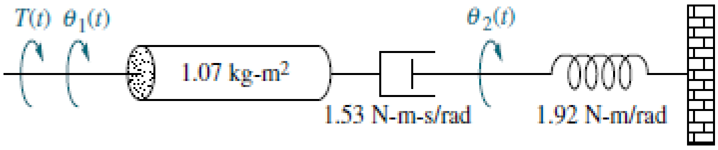




Problem #3

For the physical system shown below, a step torque is applied. Calculate:

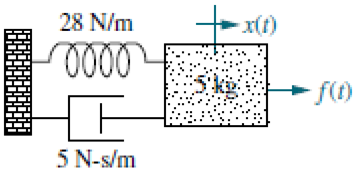
a) The system T.F. $\theta_2(s)/T(s)$
b) The peak time, max. overshoot and settling time for $\theta_2(t)$



Problem #4

For the physical system shown below, a step torque is applied. Calculate:

a) The system T.F. $X(s)/F(s)$
b) The damping ratio, undamped natural freq., rise time, peak time, max. overshoot and settling time for $x(t)$



Problem #5

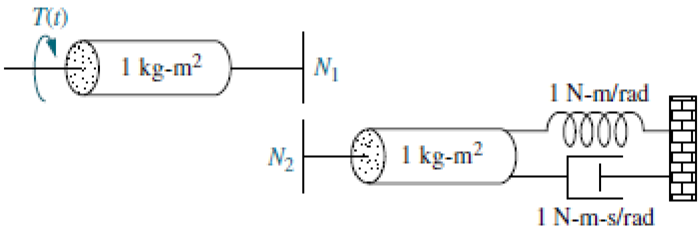
Find rise time, peak time, settling time, and percent overshoot for responses below that can be approximated as second-order responses.

$$c(t) = 0.003500 - 0.001976e^{-3t}\cos(22.16t) - 0.0005427e^{-3t}\sin(22.16t)$$

$$c(t) = 0.05100 - 0.007647e^{-6t}\cos(8t) - 0.01309e^{-6t}\sin(8t)$$

Problem #6

For the system shown below, find N_1/N_2 so that the settling time for a step torque input is 16 seconds.



Problem #7

Consider the unit-step response of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{1}{S(S+1)}$$

Obtain the rise time, peak time, maximum overshoot, and settling time.

Problem #8

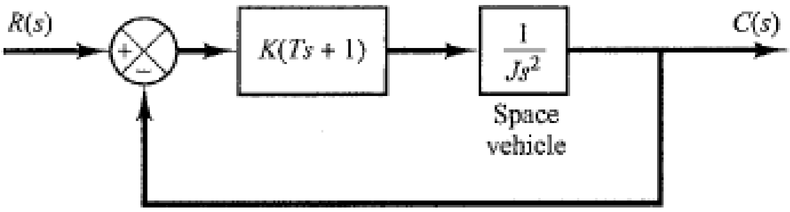
Consider the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

Determine the values of ζ and ω_n, so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec. (Use the 2% criterion.)

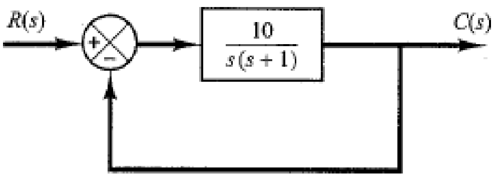
Problem #9

consider the block diagram of a space-vehicle attitude-control system given below. Assuming the time constant T of the controller to be 3 sec and the ratio K/J = 2/9 rad^2/sec^2 Find the damping ratio of the system, rise time, maximum overshoot and peak time.

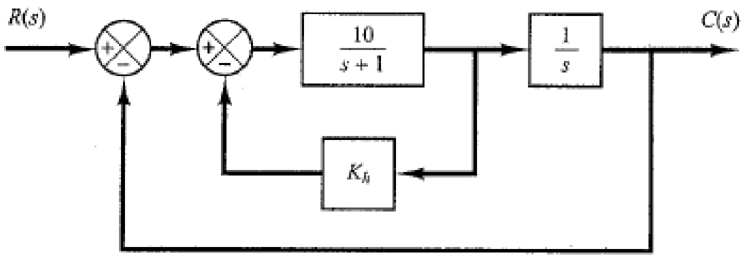


Problem #10

Consider the system shown in Fig.(a). The damping ratio of this system is 0.158 and the undamped natural frequency is 3.16 rad/sec. To improve the relative stability, we employ tachometer feedback. Fig. (b) shows such a tachometer-feedback system. Determine the value of K_h, so that the damping ratio of the system is 0.5. Then find the rise time, maximum overshoot and settling time and compare them with those obtained from the original system.



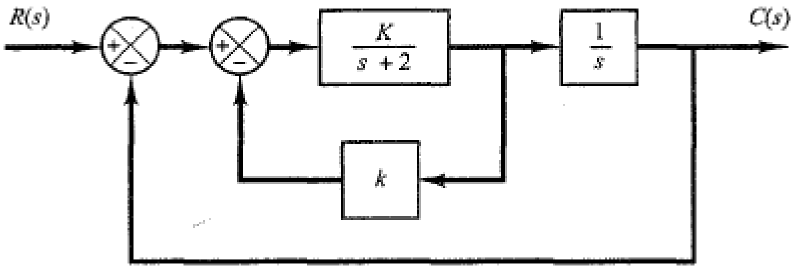
(a)



(b)

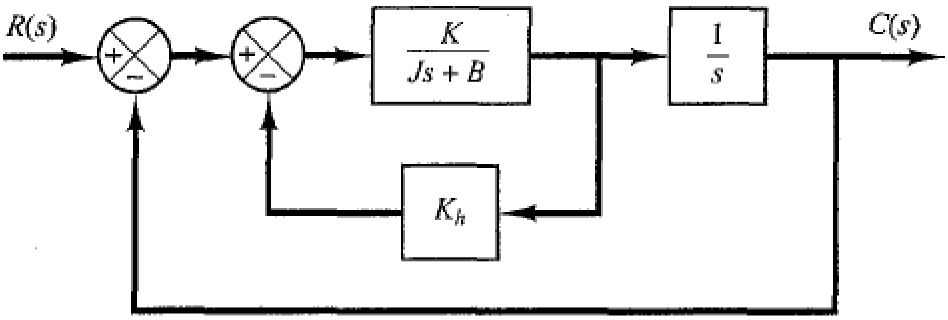
Problem #11

Referring to the system shown below, determine the values of K and k such that the system has a damping ratio 0.7 and an undamped natural of 4 rad/sec.



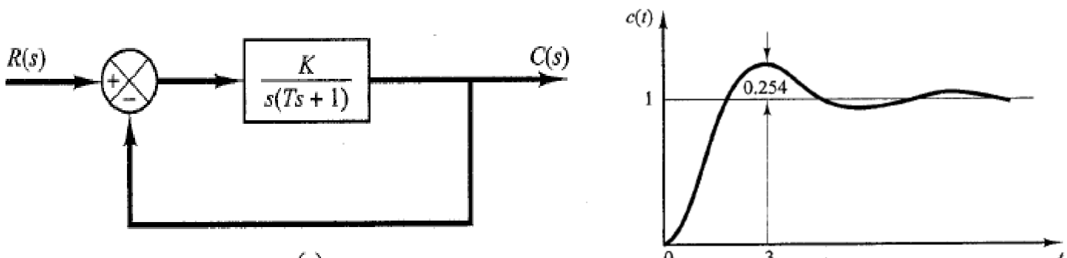
Problem #12

For the system shown below, determine the values of gain K and velocity feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time. Assume that J = 1 kg-m^2 and B = 1 N-m/rad/sec.



Problem #13

When the control system shown in Fig.(a) is subjected to a unit-step input, the system output responds as shown in Fig.(b). Determine the values of K and T from the response curve.



(a) (b)

Problem #14

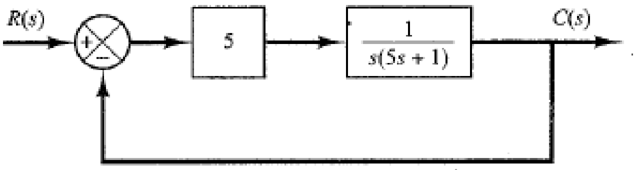
For the closed-loop system given by

$$\frac{C(s)}{R(s)} = \frac{36}{S^2 + 2S + 36}$$

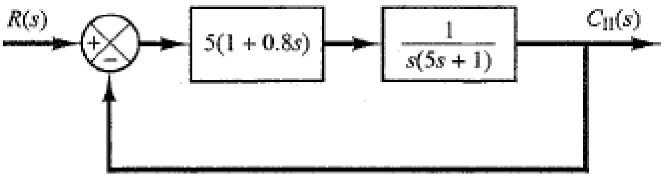
Calculate the rise time, peak time, maximum overshoot, and settling time when R(s) is considered as unit step input.

Problem #15

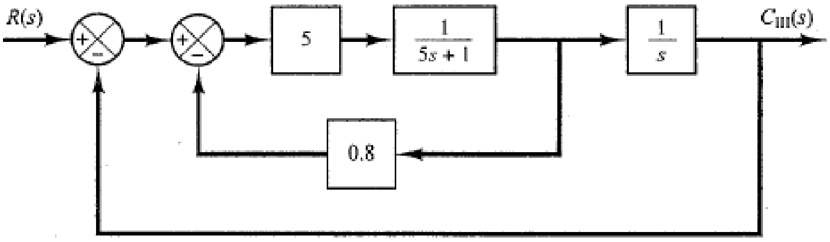
Figure below shows three systems. System I is a positional servo system. System II is a positional servo system with PD control action. System III is a positional servo system with velocity feedback. Compare the unit-step response of the three systems and obtain the best one with respect to the speed of response and maximum overshoot.



System I

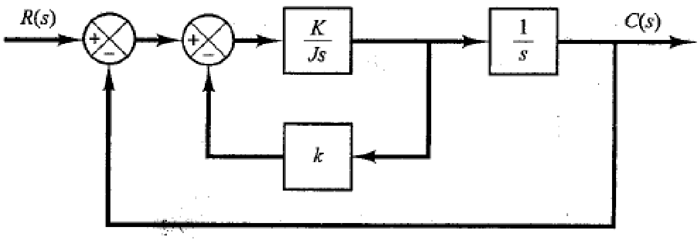


System II



System III

Problem #16



Determine the values, of $\boldsymbol{K}$ and $\boldsymbol{k}$ of the closed-loop system shown in Fig. above, so that the maximum overshoot in unit-step response is 25% and the peak time is $\boldsymbol{2}$ sec. Assume that $J = 1$ kg-m^2.

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[1] Bosch, R. GmbH. Automotive Electrics and Automotive Electronics, 5th ed. John Wiley & Sons Ltd., UK, 2007.

[2] Franklin, G. F., Powell, J. D., and Emami-Naeini, A. Feedback Control of Dynamic Systems. Addison-Wesley, Reading, MA, 1986.

[3] Dorf, R. C. Modern Control Systems, 5th ed. Addison-Wesley, Reading, MA, 1989.

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