CHAPTER 9 STABILITY ANALYSIS

After completing this chapter, the students will be able to:

- Recognize the concept of stability based on time response,
- Interpret the Routh table to check the system stability,
- Interpret the Routh table when the first element of a row is zero or when entire row is zero,
- Determine the range of gain K to guarantee stability.

1. Introduction

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system?

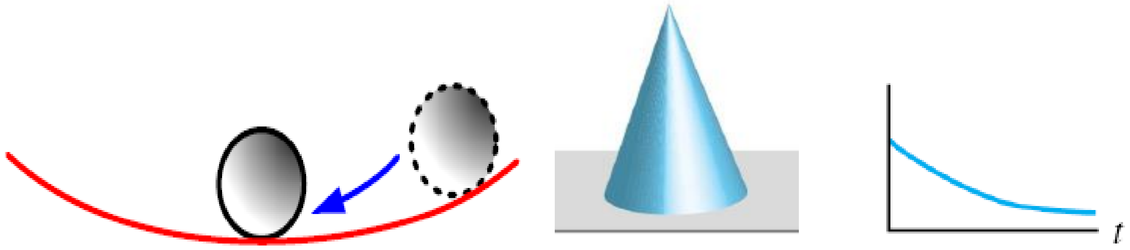
Stability may be defined as the ability of a system to restore its equilibrium position when disturbed or a system which has a bounded response for a bounded output.

Referring to Fig. 1:

(a) if the ball is displaced a small distance from this position and released, it oscillates but ultimately returns to its rest position at the base as it loses energy as a result of friction. This is therefore a *stable* equilibrium point.

(b) The stable position can be represented by a cone rest on its base.

(c) The time response of stable system converges to a certain value as the time tends to infinity.



(a) Rolling ball (b) Cone (c) time response

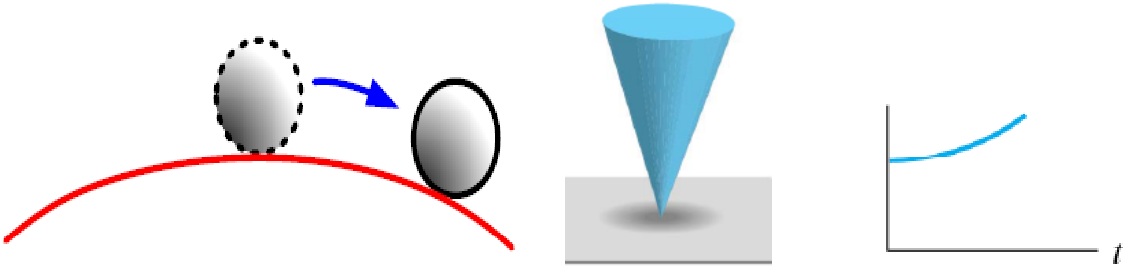
Fig. 1, Stable system

On the other hand and referring to Fig. 2:

(a) If the ball is in equilibrium as placed exactly at the top of the surface, but if it is displaced an extremely small distance to either side, the net gravitational force acting on it will cause it to roll down the surface and never return to the equilibrium point. This equilibrium is therefore *unstable*.

(b) The unstable position can be represented by a cone rest on its tip.

(c) The time response of unstable system diverges as the time tends to infinity.



(a) Rolling ball (b) Cone (c) time response

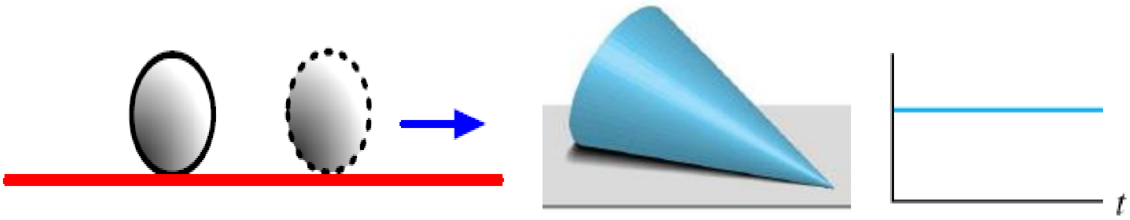
Fig. 2, Unstable system

Referring to Fig. 3:

(a) The ball neither moves away nor returns to its equilibrium position. The flat portion represents a neutrally stable region.

(b) The neutrally stable position can be represented by a cone rest on its side.

(c) The time response of neutrally stable system is constant as the time changes.



(a) Rolling ball (b) Cone (c) time response

Fig. 3, neutrally stable system

2. Stability Analysis in the Complex Plane

The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the s-plane. If any of these poles lie in the *Right-Half of the s-plane* (RHS), (either the poles are real or complex as shown in Fig. 4.) then with increasing time, they give rise to the dominant mode, and the transient response increases monotonically or oscillate with increasing amplitude. Either of these systems represents an unstable system.

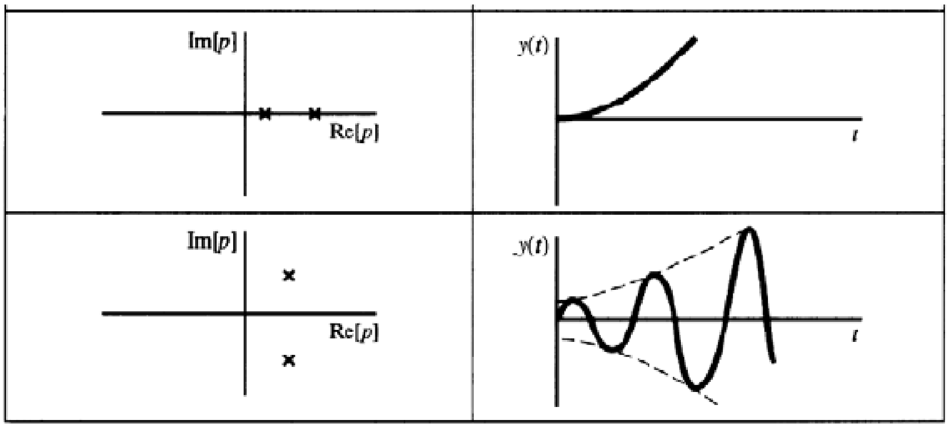


Fig. 4. Poles located in RHS gives unstable response

For such a system, as soon as the power is turned on, the output may increase with time. If no saturation takes place in the system and no mechanical stop is provided, then the system may eventually be damaged and fail, since the response of a real physical system cannot increase indefinitely.

Consider a simple feedback system shown in Fig. 5.

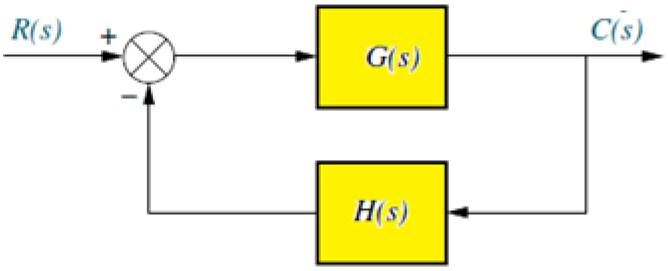


Fig. 5, closed-loop control system

The overall T.F. is given as:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation is of the above system is $1 + G(s)H(s) = 0$

The roots of the characteristic equation are called closed loop poles. The location of such roots or poles on the s-plane will indicate the condition of stability as shown in Fig. 6.

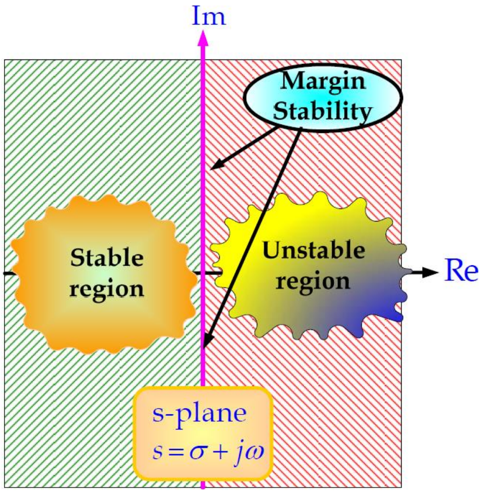
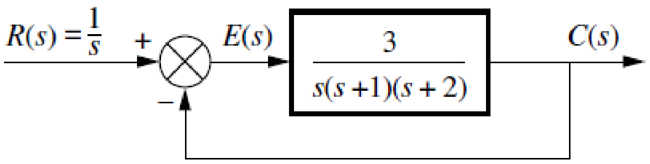
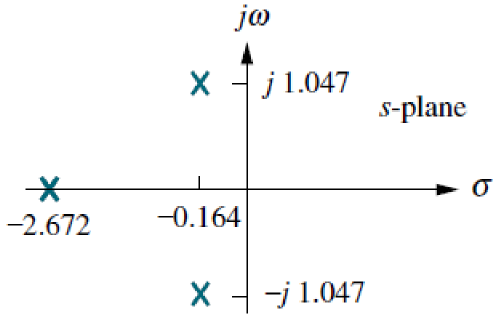


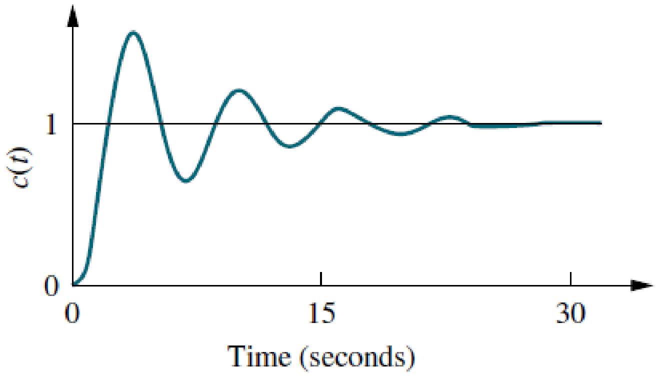
Fig. 6. Stability condition based on the location of the closed loop poles

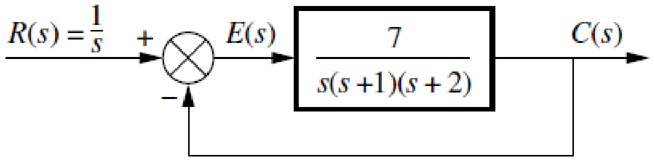


Stable system

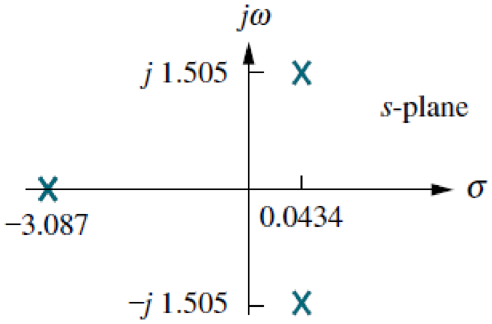


Stable system's closed-loop poles

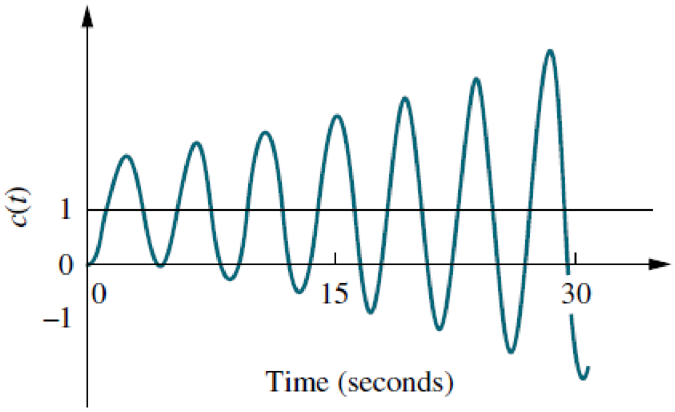




Unstable system



Unstable system's closed-loop poles



3. Routh Stability criterion (***Two Necessary but Insufficient Conditions)***

The characteristic equation of the simple feedback system can be written as a polynomial:

$$a_0S^n + a_1S^{n-1} + \cdots + a_{n-1}S^1 + a_nS^0 = 0$$

There are *two necessary but insufficient conditions* for the roots of the characteristic equation to lie in Left Hand Side (LHS) of the S-plane (i.e., stable region)

1. All the coefficients a_n, a_{n-1}, a_{n-2}, ..., a_1 *and* a_0 should have the same sign.

2. None of the coefficients vanish (All coefficients of the polynomial should exist).

By this way we judge the ***absolute stability*** of the system (stable or unstable).

Example #1

Given the characteristic equation,

$$S^6 + 4S^5 + 3S^4 - 2S^3 + S^2 + 4S + 4 = 0$$

Is the system described by this characteristic equation stable?

One coefficient (-2) is negative. Therefore, the system **does not** satisfy the necessary condition for stability. Therefore, this system is **unstable**.

Example #2

Given the characteristic equation,

$$S^6 + 4S^5 + 3S^4 + S^2 + 4S + 4 = 0$$

Is the system described by this characteristic equation stable? The term s^3 is missing. Therefore, the system **does not** satisfy the necessary condition for stability. Therefore, this system is **unstable**.

4. Hurwitz Stability Criterion (Necessary and Sufficient Condition)

In this section, we will check the system stability without the need to solve for the closed-loop system poles. Using this method, we can tell how many closed-loop poles are in LHS, in RHS, and on the jω-axis. (Notice that we say how many, not where). We can find the number of poles in each section of the s-plane, but we don't need to find their coordinates. In this method, we must arrange the coefficients of the polynomial in rows and columns according to the following pattern:

Since we are interested in the system poles, we focus on the system characteristic equation (denominator of the closed-loop T.F.) which is assumed as:

$$A_0S^n + A_1S^{n-1} + \cdots + A_{n-1}S^1 + A_nS^0 = 0$$

- Create the Hurwitz table shown in Table 1, by labelling the rows with powers of s from the highest power of the denominator (S^n) until the lowest power (S^0).

- Write the coefficient of S^n which is (A_0) and list horizontally in the first row every other even/odd coefficient (depending on *n* is even or odd, respectively).
- In the second row, list horizontally, starting with the next highest power of s which is (A_1), every coefficient that was skipped in the first row.
- The remaining entries are filled in as follows:

Table 1, Hurwitz array

S^n	A_0	A_2	A_4	A_6
S^{n-1}	A_1	A_3	A_5	A_7
S^{n-2}	$B_1 = \frac{A_1 \times A_2 - A_0 \times A_3}{A_1}$	$B_2 = \frac{A_1 \times A_4 - A_0 \times A_5}{A_1}$	$B_3 = \frac{A_1 \times A_6 - A_0 \times A_7}{A_1}$	0
S^{n-3}	$C_1 = \frac{B_1 \times A_3 - A_1 \times B_2}{B_1}$	$C_2 = \frac{B_1 \times A_5 - A_1 \times B_3}{B_1}$	$C_3 = \frac{B_1 \times A_7 - A_1 \times 0}{B_1}$	
⋮	⋮	⋮	⋮	
S^0	⋮			

Note that in developing the array an entire row may be divided or multiplied by a ***positive number*** in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Routh-Hurwitz stability criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed.

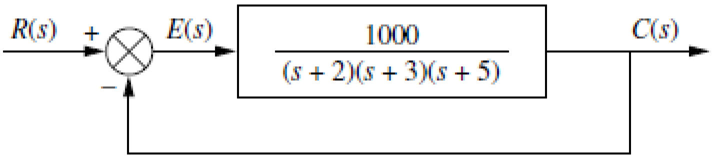
The necessary and sufficient condition that all roots of the characteristic equation lie in the left-half **s** plane is that:

a) All the coefficients of the characteristic equation be positive, and

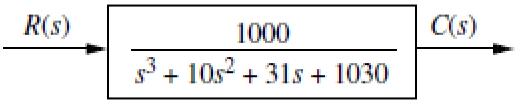
b) All terms in the first column of the array have positive signs.

Example #3

Discuss the stability of the following control system



We need to calculate the closed-loop T.F:



The system characteristic equation is:

$$S^3 + 10S^2 + 31S + 1030 = 0$$

The Hurwitz array is

(+)	S^3	1	31
(+)	S^2	10	1030
(-)	S^1	$\frac{10 \times 31 - 1 \times 1030}{10} = -72$	0
(+)	S^0	1030	

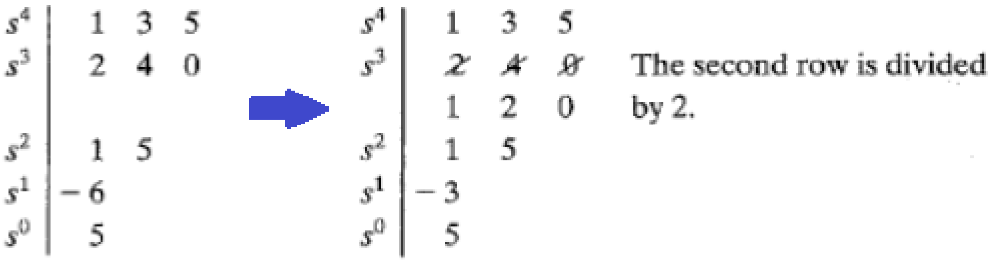
There are 2 sign changes. There are 2 poles on the RHS of the S-plane. Therefore, the system is unstable.

Example #4

Discuss the stability of the following characteristic equation:

$$S^4 + 2S^3 + 3S^2 + 4S + 5 = 0$$

Let us follow the procedure just presented and construct the array of coefficients. The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these two rows. If any coefficients are missing, they may be replaced by zeros in the array.



$$\begin{array}{c|ccc} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ & & & \\ s^2 & 1 & 5 & \\ s^1 & -6 & & \\ s^0 & 5 & & \end{array} \Rightarrow \begin{array}{c|ccc} s^4 & 1 & 3 & 5 \\ s^3 & \cancel{2} & \cancel{4} & \cancel{0} \\ & 1 & 2 & 0 \\ s^2 & 1 & 5 & \\ s^1 & -3 & & \\ s^0 & 5 & & \end{array}$$

The second row is divided by 2.

There are 2 sign changes. There are 2 poles on the right half of the S-plane. Therefore, the system is unstable.

Example #5

Check whether this system is stable or not.

$$\frac{C(s)}{R(s)} = \frac{2(S^2 + 2S + 25)}{S^5 + S^4 + 3S^3 + 9S^2 + 16S + 10}$$

The characteristic equation is:

$$S^5 + S^4 + 3S^3 + 9S^2 + 16S + 10 = 0$$

Construct Hurwitz array as follows:

S^5	1	3	16
S^4	1	9	10
S^3	–6	6	
S^2	10	10	
S^1	12		
S^0	10		

There are 2 sign changes. There are 2 poles on the right half of the S-plane. Therefore, the system is unstable.

Example #6

Check the stability of the control system whose characteristic equation is:

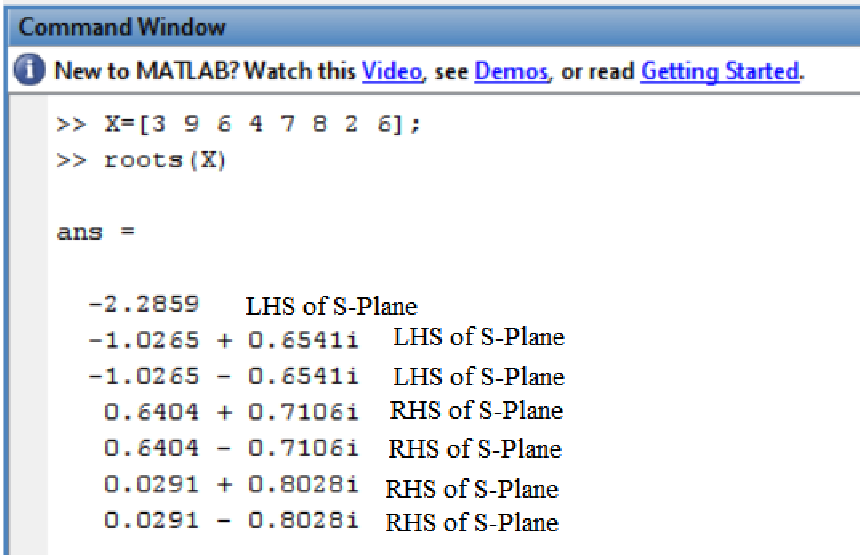
$$3S^7 + 9S^6 + 6S^5 + 4S^4 + 7S^3 + 8S^2 + 2S + 6 = 0$$

Construct Hurwitz array as follows:

S^7	3	6	7	2
S^6	9	4	8	6
S^5	4.6667	4.3333	0	
S^4	-4.357	8	6	
S^3	12.90165	6.4265		
S^2	10.1703	6		
S^1	-1.1849			
S^0	6			

There are 4 sign changes. There are 4 poles on the right half of the S-plane. Therefore, the system is unstable.

From Matlab, we can obtain the roots of this characteristic equation. It is clear that there are 3 roots lie in the LHS of S-plane and there are 4 roots lie in the RHS of S-plane.



```
>> X=[3 9 6 4 7 8 2 6];
>> roots(X)

ans =

  -2.2859             LHS of S-Plane
  -1.0265 + 0.6541i   LHS of S-Plane
  -1.0265 - 0.6541i   LHS of S-Plane
   0.6404 + 0.7106i   RHS of S-Plane
   0.6404 - 0.7106i   RHS of S-Plane
   0.0291 + 0.8028i   RHS of S-Plane
   0.0291 - 0.8028i   RHS of S-Plane
```

5. Special Cases for Hurwitz Array:

1- If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number **ε** and the rest of the array is evaluated.

Example #7

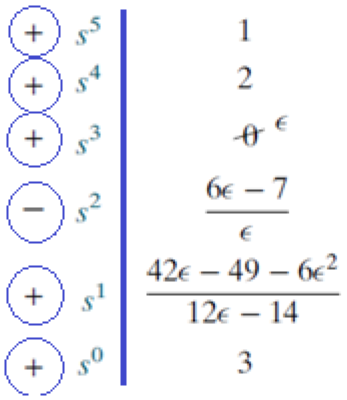
Determine the stability of the closed-loop system given below.

$$\frac{C(s)}{R(s)} = \frac{10}{S^5 + 2S^4 + 3S^3 + 6S^2 + 5S + 3}$$

The system characteristic equation is:

$$S^5 + 2S^4 + 3S^3 + 6S^2 + 5S + 3 = 0$$

The Hurwitz array is: (note that, we replace the "0" of S^3 row by ε)



⊕	s^5	1	3	5
⊕	s^4	2	6	3
⊕	s^3	$\cancel{0}\ \epsilon$	$\frac{7}{2}$	
⊖	s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	
⊕	s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$		
⊕	s^0	3		

In each case (either ε is +ve or -ve) the system is unstable with two poles located at the RHS of S-plane.

Another Solution:

The original characteristic equation is:

$$S^5 + 2S^4 + 3S^3 + 6S^2 + 5S + 3 = 0$$

Form a polynomial that has the reciprocal order of the characteristic equation:

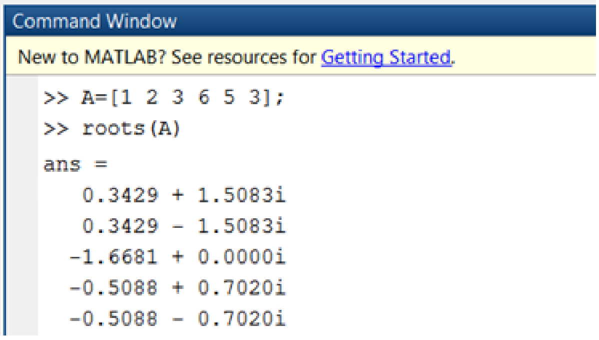
$$3S^5 + 5S^4 + 6S^3 + 3S^2 + 2S + 1 = 0$$

Construct Hurwitz array as follows:

S^5	3	6	2
S^4	5	3	1
S^3	4.2	1.4	
S^2	1.3333	1	
S^1	-1.75		
S^0	1		

the system is unstable with two poles located at the RHS of S-plane as result obtain in case of ε used.

We can check the answer by factorizing the characteristic eqn. by Matlab:



```
>> A=[1 2 3 6 5 3];
>> roots(A)
ans =
   0.3429 + 1.5083i
   0.3429 - 1.5083i
  -1.6681 + 0.0000i
  -0.5088 + 0.7020i
  -0.5088 - 0.7020i
```

Example #8

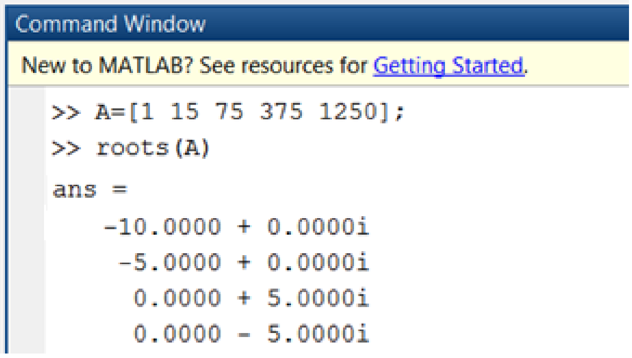
Consider the following characteristic equation:

$$S^4 + 15S^3 + 75S^2 + 375S + 1250 = 0$$

S^4	1	75	1250
S^3	15	375	
S^2	50	1250	
S^1	ε ~~0~~		
S^0	1250		

If the sign of the coefficient above the zero ε is the same as that below ε, it indicates that there is a pair of imaginary roots. Therefore, the system is marginally stable as there are no sign changes. To get the poles located at the jω axis: go directly to the raw over the 0 cell and form the equation: $50\,S^2 + 1250 = 0 \rightarrow S = \pm J\,5$

This can be obtained by finding the roots by Matlab



```
>> A=[1 15 75 375 1250];
>> roots(A)
ans =
  -10.0000 + 0.0000i
   -5.0000 + 0.0000i
    0.0000 + 5.0000i
    0.0000 - 5.0000i
```

2- If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the ***s*** plane, that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.

In such a case, the evaluation of the rest of the array can be continued by forming an ***auxiliary polynomial*** with the coefficients of the last row and by using the coefficients of the ***derivative of this auxiliary*** polynomial in the next row. Such roots with equal magnitudes and lying radially opposite in the ***s*** plane can be found by solving the auxiliary polynomial, which is always ***even***.

Example #9:

Determine the stability of the closed-loop system given below, and find the location of the closed-loop poles.

$$\frac{C(s)}{R(s)} = \frac{10}{S^5 + 7S^4 + 6S^3 + 42S^2 + 8S + 56}$$

The system characteristic equation is:

$$S^5 + 7S^4 + 6S^3 + 42S^2 + 8S + 56 = 0$$

Construct Hurwitz array:

$$\begin{array}{c|ccc} s^5 & 1 & 6 & 8 \\ s^4 & \cancel{7}\ 1 & \cancel{42}\ 6 & \cancel{56}\ 8 \\ s^3 & 0 & 0 & \end{array}$$

Since all row coefficients are zeros, to solve this problem: First we return to the row immediately above the row of zeros and form an auxiliary equation A(s), using the entries in that row as coefficients. The polynomial will start with the power of s in the label column and continue by skipping every other power of s. Thus, the polynomial formed for this example is

$$A(s) = S^4 + 6S^2 + 8$$

Next, we differentiate the equation with respect to s and obtain

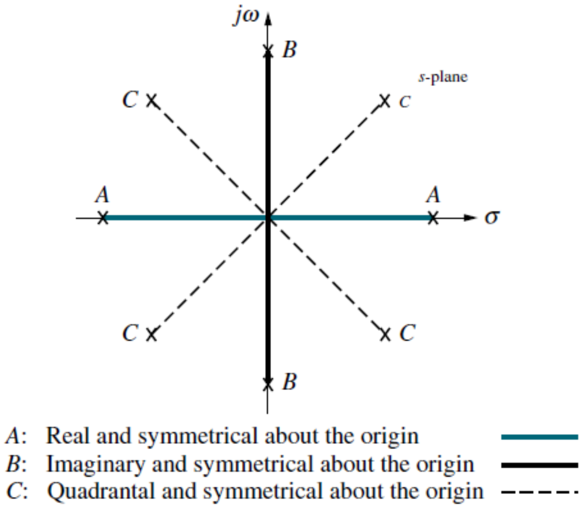
$$\frac{dA(s)}{dS} = 4S^3 + 12S$$

Finally, we use the coefficients of that derivative to replace the row of zeros.

$$\begin{array}{c|ccc}
s^5 & 1 & 6 & 8 \\
s^4 & 1 & 6 & 8 \\
s^3 & \cancel{4}\ 1 & \cancel{12}\ 3 & \\
s^2 & 3 & 8 & \\
s^1 & \frac{1}{3} & & \\
s^0 & 8 & &
\end{array}$$

From the first column sign, all entries in the first column are positive. Hence, there are no right–half-plane poles.

An entire row of zeros will appear in the Routh table when a purely even polynomial is a factor of the original polynomial. As we see the auxiliary equation A(s) is an even polynomial; it has only even powers of s. Even polynomials only have roots that are symmetrical about the origin. This symmetry can occur under three conditions of root position: (1) The roots are symmetrical and real, (2) the roots are symmetrical and imaginary, or (3) the roots are quadrantal. See Figure below.



Solve for the roots of the Auxiliary eqn.:

$$A(s) = S^4 + 6S^2 + 8 = 0$$

Assuming that S^2 by x,

$$A(x) = x^2 + 6x + 8 = 0$$

$$x = -2 \quad \to S_{1,2} = \pm J\sqrt{2}$$

$$x = -4 \quad \to S_{3,4} = \pm J2$$

This means there 4 poles on jω axis, and 1 pole in the LHS of s-plane. Therefore, the system is **Marginally (Critically)** **stable** system.

Example #10

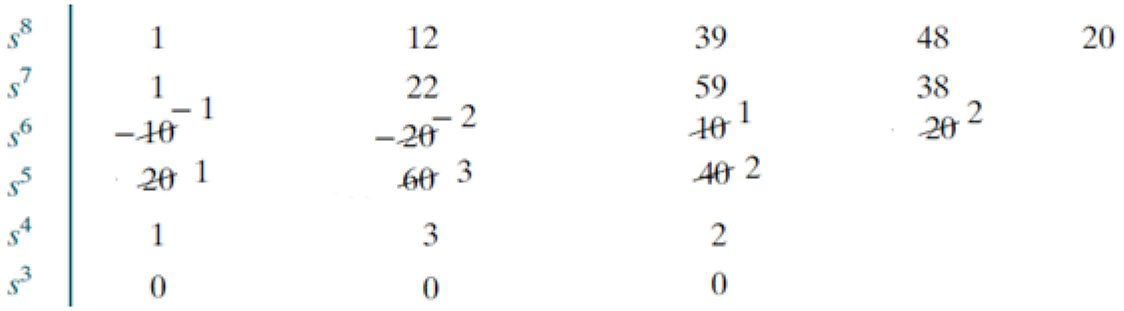
Determine the stability of the closed-loop system given below, and find the location of the closed-loop poles.

$$\frac{C(s)}{R(s)} = \frac{20}{S^8 + S^7 + 12S^6 + 22S^5 + 39S^4 + 59S^3 + 48S^2 + 38S + 20}$$

The system characteristic equation is:

$$S^8 + S^7 + 12S^6 + 22S^5 + 39S^4 + 59S^3 + 48S^2 + 38S + 20 = 0$$

Construct Hurwitz array:



s^8	1	12	39	48	20
s^7	1	22	59	38	
s^6	$-\cancel{10}\ -1$	$-\cancel{20}\ -2$	$\cancel{10}\ 1$	$\cancel{20}\ 2$	
s^5	$\cancel{20}\ 1$	$\cancel{60}\ 3$	$\cancel{40}\ 2$		
s^4	1	3	2		
s^3	0	0	0		

we return to the row immediately above the row of zeros and form an auxiliary equation A(s),

$$A(s) = S^4 + 3S^2 + 2$$

The derivative will be

$$\frac{dA(s)}{dS} = 4S^3 + 6S$$

Replace the zeros by the coefficient of above equation

s^8	1	12	39	48	20
s^7	1	22	59	38	
s^6	-1	-2	1	2	
s^5	1	3	2		
s^4	1	3	2		
s^3	2	3			
s^2	3	4			
s^1	$\frac{1}{3}$				

Solve for the roots of the Auxiliary eqn.:

$$A(s) = S^4 + 3S^2 + 2 = 0$$

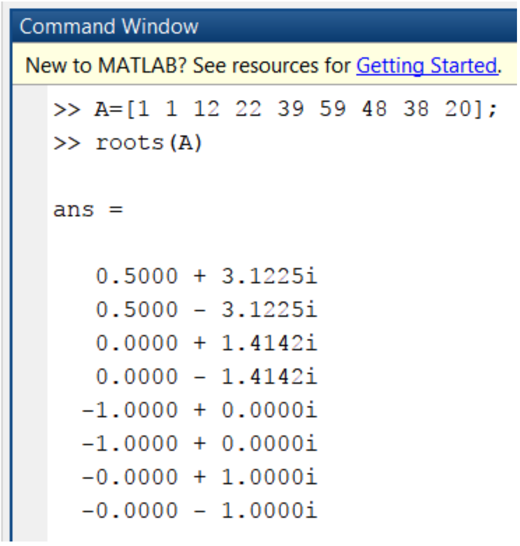
Assuming that S^2 by *x*,

$$A(x) = x^2 + 3x + 2 = 0$$

$$x = -2 \quad \to S_{1,2} = \pm J\sqrt{2}$$

$$x = -1 \quad \to S_{3,4} = \pm J1$$

Thus, the system has two poles in the right half-plane, two poles in the left half-plane, and four poles on the jω-axis; therefore, the system is unstable because of the right–half-plane poles. This is clear also from Matlab roots of that characteristic equation:



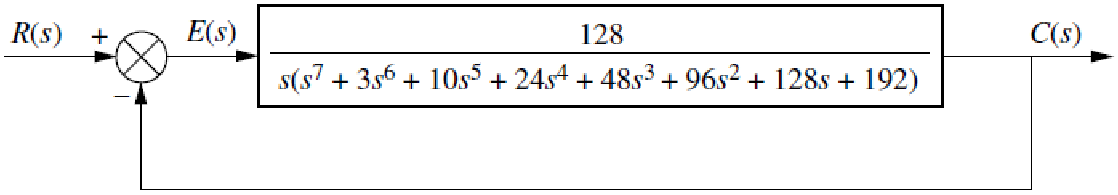
```
>> A=[1 1 12 22 39 59 48 38 20];
>> roots(A)

ans =

   0.5000 + 3.1225i
   0.5000 - 3.1225i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -1.0000 + 0.0000i
  -1.0000 + 0.0000i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

Example #11:

Find the number of poles in the left half-plane, the right half-plane, and on the jω-axis for the system of figure below. Draw conclusions about the stability of the closed-loop system.



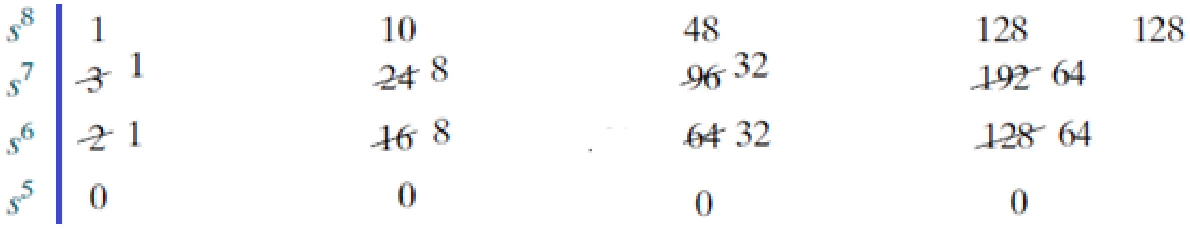
The closed-loop T.F. is given by:

$$\frac{C(s)}{R(s)} = \frac{128}{S^8 + 3S^7 + 10S^6 + 24S^5 + 48S^4 + 96S^3 + 128S^2 + 192S + 128}$$

The system characteristic equation is:

$$S^8 + 3S^7 + 10S^6 + 24S^5 + 48S^4 + 96S^3 + 128S^2 + 192S + 128$$

Using Routh-Hurwitz array,



s^8	1	10	48	128	128
s^7	~~3~~ 1	~~24~~ 8	~~96~~ 32	~~192~~ 64	
s^6	~~2~~ 1	~~16~~ 8	~~64~~ 32	~~128~~ 64	
s^5	0	0	0	0	

Since there is a row with all coefficients are zeros;

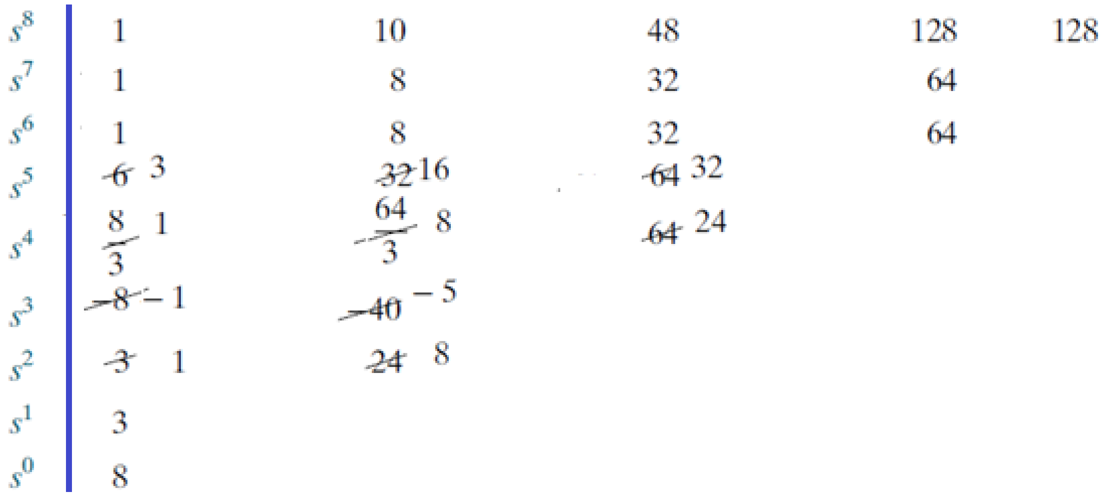
we return to the row immediately above the row of zeros and form an auxiliary equation A(s),

$$A(s) = S^6 + 8S^4 + 32S^2 + 64$$

The derivative will be:

$$\frac{dA(s)}{dS} = 6S^5 + 32S^3 + 64S$$

Replace the zeros by the coefficient of above equation



s^8	1	10	48	128	128
s^7	1	8	32	64	
s^6	1	8	32	64	
s^5	~~6~~ 3	~~32~~ 16	~~64~~ 32		
s^4	$\cancel{\frac{8}{3}}$ 1	$\cancel{\frac{64}{3}}$ 8	~~64~~ 24		
s^3	~~-8~~ −1	~~-40~~ −5			
s^2	~~3~~ 1	~~24~~ 8			
s^1	3				
s^0	8				

Since there are two sign change, there two poles located at right-half plane.

Solve for the roots of the Auxiliary eqn.:

$$A(s) = S^6 + 8S^4 + 32S^2 + 64 = 0$$

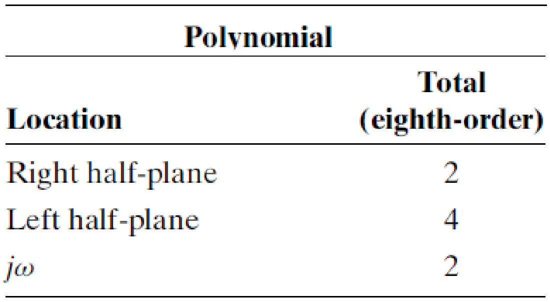
Assuming that S^2 by x,

$$A(x) = x^3 + 8x^2 + 32x + 64 = 0$$

$x = -4 \rightarrow S_{1,2} = \pm J2$ → two pure imaginary poles

$x = -2 + J3.4641 \rightarrow S_{3,4} = \pm(-1 + J1.73205) \rightarrow$ two poles located in LHS

$x = -2 - J3.4641 \rightarrow S_{3,4} = \pm(-1 - J1.73205) \rightarrow$ two poles located in LHS



Polynomial	
Location	**Total (eighth-order)**
Right half-plane	2
Left half-plane	4
$j\omega$	2

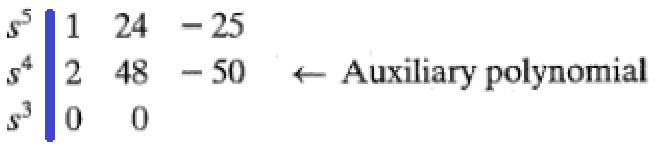
Example #12:

Consider the following characteristic equation:

$$S^5 + 2S^4 + 24S^3 + 48S^2 - 25S - 50 = 0$$

Discuss the system stability and find the location of closed-loop poles.

Construct the Routh-Hurwitz array as:



$$\begin{array}{c|ccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array} \quad \leftarrow \text{Auxiliary polynomial}$$

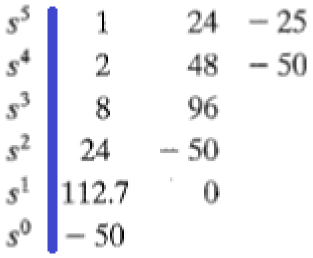
The terms in the **s^3** row are all zero. (Note that such a case occurs only in an odd numbered row.) The auxiliary polynomial is then formed from the coefficients of the **s^4** row. The auxiliary polynomial ***A(s)*** is

$$A(s) = 2S^4 + 48S^2 - 50$$

The derivative will be:

$$\frac{dA(s)}{dS} = 8S^3 + 96S$$

Replace the zeros by the coefficient of above equation



$$\begin{array}{c|ccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 8 & 96 & \\ s^2 & 24 & -50 & \\ s^1 & 112.7 & 0 & \\ s^0 & -50 & & \end{array}$$

We see that there is one change in sign in the first column of the new array. Thus, the characteristic equation has one pole with a positive real part. By solving for roots of the auxiliary equation,

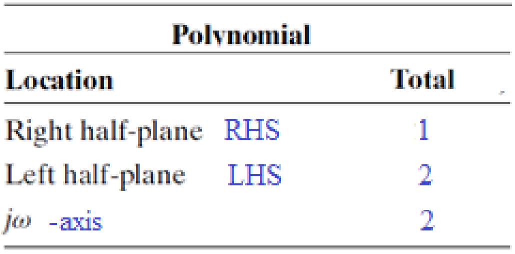
$$2S^4 + 48S^2 - 50 = 0$$

Assuming that S^2 by x,

$$A(x) = x^2 + 24x - 25 = 0$$

$x = 1 \ \rightarrow S_{1,2} = \pm 1$ → one pole in RHS & one pole in LHS

$x = -25 \ \rightarrow S_{3,4} = \pm J5$ → two pure imaginary poles



Polynomial		
Location		**Total**
Right half-plane	RHS	1
Left half-plane	LHS	2
$j\omega$ -axis		2

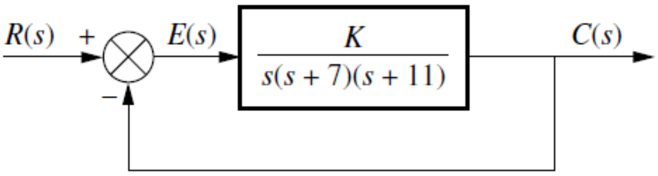
6. Stability Design using Routh-Hurwitz Criterion

The Routh-Hurwitz criterion gives a strong proof that changes in the gain of a feedback control system result in differences in transient response because of changes in closed-loop pole locations.

It is possible to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Example #13:

Find the range of gain, K, for the system shown below that will cause the system to be stable, then find the frequency of sustained oscillation..



$$\frac{C(s)}{R(s)} = \frac{K}{S^3 + 18S^2 + 77S + K}$$

The system characteristic equation is:

$$S^3 + 18S^2 + 77S + K = 0$$

Using Routh-Hurwitz array,

$$\begin{array}{c|cc} s^3 & 1 & 77 \\ s^2 & 18 & K \\ s^1 & \dfrac{1386 - K}{18} & \\ s^0 & K & \end{array}$$

From the above array

From S^0 raw $K > 0$ (1)

From S^2 raw $1386 - K > 0 \rightarrow K < 1386$ (2)

Then the range of gain K for stability is

$$1386 > K > 0$$

At K=1386 exactly, the system becomes oscillatory (critically stable) and the oscillation is sustained at constant amplitude. To get this frequency we form the auxiliary equation A(S) from the coefficients of the row above that contain K=1386

$$A(S) = (18)\,S^2 + (1386) = 0$$

Solving this equation to get the frequency.

$$S = \pm J8.775\ rad/s$$

Example #14

Consider the system described by a closed-loop transfer function given below, and we need to determine the range of K for stability.

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

$$\begin{array}{llll}
s^4 & 1 & 3 & K \\
s^3 & 3 & 2 & 0 \\
s^2 & \frac{7}{3} & K & \\
s^1 & 2 - \frac{9}{7}K & & \\
s^0 & K & &
\end{array}$$

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

At $K = \frac{14}{9}$ the system becomes oscillatory (critically stable) and the oscillation is sustained at constant amplitude. To get this frequency we form the auxiliary equation A(S) from the coefficients of the row above that contain $K = \frac{14}{9}$

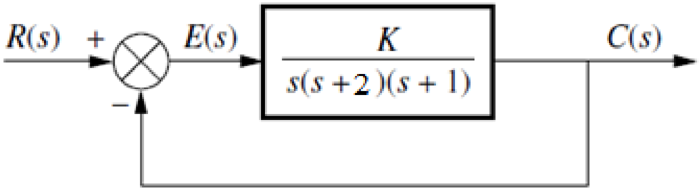
$$A(S) = (7/3)\,S^2 + (14/9) = 0$$

Solving this equation to get the frequency.

$$S = \pm J0.8165\ rad/s$$

Example #15

In the figure below, determine the range of *K* for the system to be stable



The characteristic equation is:

$$S^3 + 3S^2 + 2S + K = 0$$

Construct Routh array

$$\begin{array}{c|cc} S^3 & 1 & 2 \\ S^2 & 3 & K \\ S^1 & \dfrac{6-K}{3} & 0 \\ S^0 & K & \end{array}$$

For the system stability;

From the S^0 row K > 0,

From S^1 row, 6 − K > 0 → K < 6

Then for stability, 6 > K > 0

at *K* = 6, the above characteristic equation becomes

$$S^3 + 3S^2 + 2S + 6 = 0$$

$$\Rightarrow \begin{cases} s = -3 & \\ s = +j\sqrt{2} & \Rightarrow \text{The system is marginally stable} \\ s = -j\sqrt{2} & \Rightarrow \text{The system is marginally stable} \end{cases}$$

The frequency of oscillations in the previous case is √2 rad/s.

Or at K=6, the auxiliary equation is:

$3S^2 + 6 = 0 \rightarrow S^2 = -2 \rightarrow S = \pm j\sqrt{2}$

Example #16:

The open-loop transfer function of a control system may be approximated by

$$GH(S) = \frac{K(S+10)}{S(S+3)(S^2+4S+8)}$$

- Determine the range of gain K for the stability of the system,
- Calculate the maximum value of K for stability and the frequency of oscillation.

The system characteristic equation is

$S(S+3)(S^2+4S+8) + K(S+10) = 0$

$S^4 + 7\,S^3 + 20\,S^2 + (24+K)\,S + 10\,K = 0$

Using Routh array

S^4	1	20	10K
S^3	7	24+K	
S^2	116 – K	70 K	
S^1	$\frac{-(K^2+398K-2784)}{116-K}$		
S^0	70 K		

From the above array

From S^0 raw $\quad 70K > 0 \quad \rightarrow \quad K > 0 \quad (1)$

From S^2 raw $\quad 116 - K > 0 \quad \rightarrow \quad K < 116 \quad (2)$

From S^1 raw $\quad -(K^2+398K-2784) > 0 \quad \rightarrow \quad K^2+398K-2784 < 0$

$$(K-6.8762)(K+404.876) < 0$$

In that case we have 2 scenarios:

First Scenario

$(K-6.8762) > 0 \ \rightarrow K > 6.8762$

$(K+404.876) < 0 \ \rightarrow K < -404.876$

Second Scenario

$(K-6.8762) < 0 \ \rightarrow K < 6.8762 \quad (3)$

$(K+404.876) > 0 \ \rightarrow K > -404.876 \quad (4)$

Due to condition 1 & 2, we accept the Second Scenario

From conditions (1) & (4): K > 0

From conditions (2) & (3): K< 6.8762

The range of K for stability is

$$0 < K < 6.8762$$

The maximum value of K for stability is at K = 6.8762

This value is obtained from the row S^1 so the auxiliary equation A(S) can be obtained from S^2

$A(S) = (116 – K) S^2 + 70K = 0$

$109.1238 S^2 + 481.334 = 0$

$S = \pm j 2.1$

The frequency of continuous oscillation is 2.1 rad/sec.

Example #17:

A simplified form of open-loop, unity-feedback transfer function of an airplane with an autopilot in the longitudinal mode is

$$G(S)H(S) = \frac{K(S+1)}{S(S-1)(S^2+4S+16)}$$

Such system involving an open-loop pole in the right-half S plane may be conditionally stable.

a) Using Routh-Hurwitz criteria, find the range of K for stability.
b) Calculate the corresponding frequency of sustained oscillation.
c) Determine the system type.
d) For the value (values) of K obtained in (a), and for unit ramp input, calculate the system steady-state error.

Since the system is unity feedback.

Then the system characteristic equation can be obtained by adding the numerator and denominator of G(S)H(S)

$$S^4 + 3S^3 + 12S^2 + (K-16)S + K = 0$$

Using Routh-Hurwitz array

S^4	1	12	K
S^3	3	K–16	
S^2	$\frac{52-K}{3}$	K	
S^1	$\frac{-K^2+59K-832}{52-K}$		
S^0	K		

From the above array:

From S^0 raw K>0 (1)

From S^2 raw 52–K>0 → K < 52 (2)

From S^2 raw $-K^2 + 59K - 832 > 0 \rightarrow K^2 - 59K + 832 < 0$

The values of K that make the s^1 term in the first column equal zero are

K = *35.685* and *K* = *23.315*.

$$(K - 35.685)(K - 23.315) < 0$$

In that case we have 2 scenarios:

First Scenario	Second Scenario
$(K - 35.685) > 0 \rightarrow K > 35.685$	$(K - 35.685) < 0 \rightarrow K < 35.685$ (3)
$(K - 23.315) < 0 \rightarrow K < 23.315$	$(K - 23.315) > 0 \rightarrow K > 23.315$ (4)

Due to condition 1 & 2, we accept the Second Scenario

From conditions (1) & (4): K > 23.315

From conditions (2) & (3): K< 35.685

Then the range of K is

$$23.315 < K < 35.685$$

The crossing points on the imaginary axis can be found by solving the auxiliary equation obtained from the s^2 row, that is, by solving the following equation for *s:*

$$\frac{52 - K}{3} s^2 + K = 0$$

The results are

$$S = \pm J2.56 \qquad at\ K = 35.685$$

$$S = \pm J1.56 \qquad at\ K = 23.315$$

The system type is 1

For unit ramp input, we calculate the velocity error coefficient K_v

K_v = lim (S→0) SG(S)H(S)

$$K_v = \frac{K}{-1 \times 16} = -\frac{35.7}{16}\ Or - \frac{23.3}{16}$$

$E_{ss} = 1/K_v$

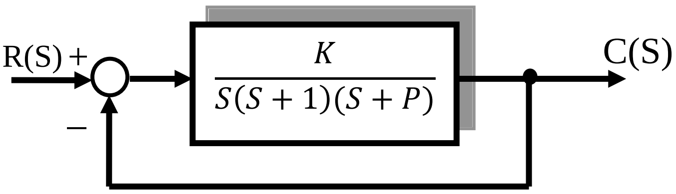
$$E_{ss} = -\frac{16}{35.7} = -0.44818$$
$$E_{ss} = -\frac{16}{23.3} = -6867$$

Example #18:

It is found that, the unity feedback control system shown in Fig. 3, is stable for the range $0 \le K \le 2.0$

a) Determine the value of P to fulfill this condition

b) Calculate the frequency of sustained oscillation.



The system characteristic equation is given by:

$S^3 + (P+1)\, S^2 + P\, S + K = 0$

Using Routh,

$$\begin{array}{l|ll} S^3 & 1 & P \\ S^2 & P+1 & K \\ S & \frac{P^2+P-K}{P+1} & \\ S^0 & K & \end{array}$$

a) For stability

$K > 0$

$P^2 + P - K > 0 \rightarrow K < P^2 + P$

Then $0 < K < P^2 + P$

But the stability condition is given as $0 < K < 2$

By comparing we get

$P^2 + P = 2 \rightarrow P^2 + P - 2 = 0$

$(P+2)(P-1) = 0$

$P = 1$ OR $P = -2$

But in the 2nd row of routh array, there is a stability condition P+1 must be +ve

So if P = -2, this mean the value of P+1 = -1 which is negative

Therefore P = -2 is rejected

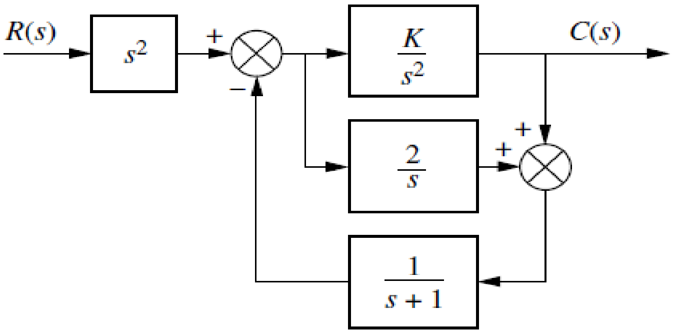
P = 1 ###

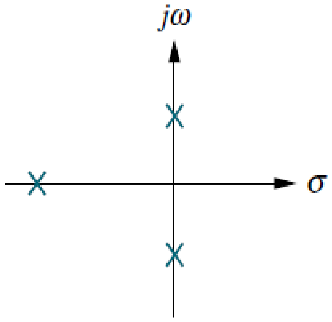
b) The auxiliary equation $A(S) = 2S^2 + 2 = 0$

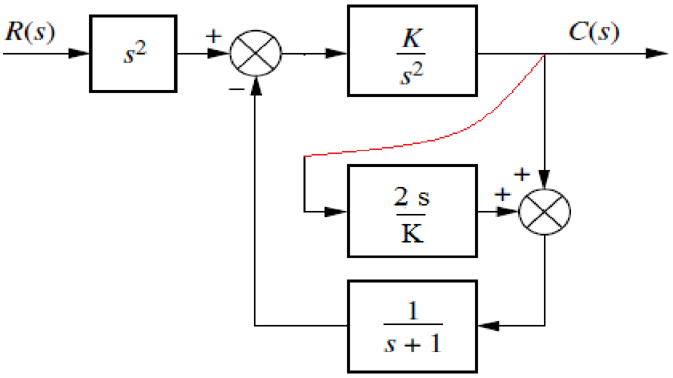
S = ± J 1 rad/s which is the frequency of sustained oscillation ###

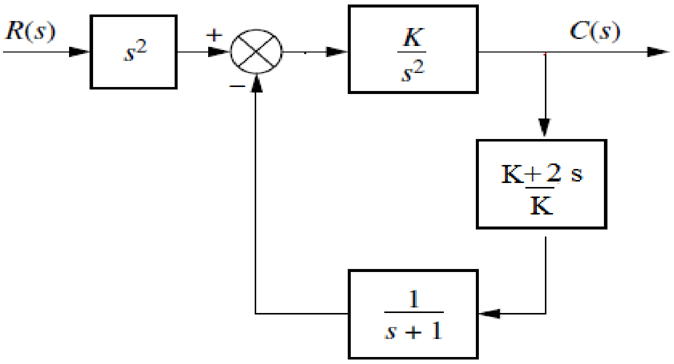
Example #19:

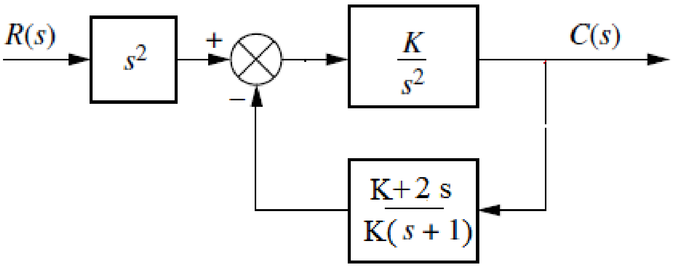
Find the value of K in the system given in figure below that will place the closed-loop poles as shown and find the value of each pole.











$$\frac{C(s)}{R(s)} = \frac{K^2S^2(S+1)}{KS^2(S+1)+K^2+2KS}$$

$$\frac{C(s)}{R(s)} = \frac{KS^2(S+1)}{S^2(S+1)+K+2S}$$

Therefore, the system characteristic equation is:

$S^3 + S^2 + 2S + K = 0$

Using Routh array;

$$\begin{array}{l|ll} S^3 & 1 & 2 \\ S^2 & 1 & K \\ S & 2-K & \\ S^0 & 2 & \end{array}$$

For the system to be marginally stable K=2

we return to the characteristic equation at K=2 and solve it,

$S^3 + S^2 + 2S + 2 = 0$

Solving this equation gives the three roots at:

$S_1 = +j\ 1.4142$

$S_2 = -j\ 1.4142$

$S_3 = -2$

Example #20:

The open-loop transfer function of a control system may be approximated by

$$GH(S) = \frac{K(S+3)}{S(S+5)(S+6)(S^2+2S+2)}$$

- Determine the range of gain K for the stability of the system,
- Calculate the maximum value of K for stability and the frequency of oscillation.

The system characteristic equation is:

$$S^5 + 13\,S^4 + 54\,S^3 + 82\,S^2 + (60+K)\,S + 3\,K = 0$$

Using Routh array

$$\begin{array}{l|ccc} S^5 & 1 & 54 & 60+K \\ S^4 & 13 & 82 & 3K \\ S^3 & \dfrac{620}{13} & \dfrac{780+10K}{13} & \\ S^2 & \dfrac{3130.7692 - 10K}{47.69231} & 3K & \\ S^1 & \dfrac{-(100K^2+65200K-2441999.976)}{3130.7692 - 10K} & & \\ S^0 & 3K & & \end{array}$$

From the above array

From S^0 raw $3K > 0$ → $K > 0$ (1)

From S^2 raw $3130.7692 - 10K > 0 \rightarrow K < 313.07692$ (2)

From S^1 raw $-(K^2 + 652K - 24420) > 0 \rightarrow K^2 + 652K - 24420 < 0$

$$(K - 35.519)(K + 687.519) < 0$$

In that case we have 2 scenarios:

First Scenario	Second Scenario
$(K - 35.519) > 0 \rightarrow K > 35.519$	$(K - 35.519) < 0 \rightarrow K < 35.519$ (3)
$(K + 687.519) < 0 \rightarrow K < -687.519$	$(K + 687.519) > 0 \rightarrow K > -687.519$ (4)

Due to condition 1 & 2, we accept the Second Scenario

From conditions (1) & (4): $K > 0$

From conditions (2) & (3): $K < 35.519$

The range of K for stability is

$$0 < K < 35.519$$

The maximum value of K for stability is at $K = 35.519$

This value is obtained from the row S^1 so the auxiliary equation A(S) can be obtained from S^2

$$A(s) = \frac{3130.7692 - 10K}{47.69231} S^2 + 3K = 0$$

$$A(s) = 58.19763S^2 + 106.557 = 0$$

$$S = \pm j\, 1.353$$

Example #21:

The open-loop transfer function of a unity-feedback control system may be approximated as:

$$G(s)H(s) = \frac{K(S^2 + 2S + 4)}{S^5 + 11.4S^4 + 39S^3 + 43.6S^2 + 24S}$$

(a) Using Routh-Hurwitz, determine the range of gain K for stability,

(b) At the maximum values of the range obtained in (a), the system oscillates continuously, calculate the frequency of this oscillation,

The system characteristic equation is:

$$S^5 + 11.4S^4 + 39S^3 + 43.6S^2 + 24S + K(S^2 + 2S + 4) = 0$$

$$S^5 + 11.4S^4 + 39S^3 + (43.6 + K)S^2 + (24 + 2K)S + 4K = 0$$

Using Routh array

S^5	1	39	24+2K
S^4	11.4	43.6+K	4K
S^3	401 – K	273.6+18.8K	
S^2	$\frac{-K^2+143.08K+14364.56}{401-K}$	4K	
S^1	$\frac{-22.8K^3+5624.304K^2-334003.584K+3930143.616}{-K^2+143.08K+14364.56}$		
S^0	4 K		

a) From the above array

from row S^0: $4K > 0$ → $K > 0$ (1)

from row S^3: $401 - K > 0$ → $K < 401$ (2)

from row S^2: $-K^2 + 143.08K + 14364.56 > 0$

$$K^2 - 143.08K - 14364.56 < 0$$

$$(K - 211.1198)(K + 68.0398) < 0$$

In that case we have 2 scenarios:

First Scenario

$(K - 211.1198) > 0 \ \rightarrow K > 211.1198$

$(K + 68.0398) < 0 \ \rightarrow K < -68.0398$

Second Scenario

$(K - 211.1198) < 0 \ \rightarrow K < 211.1198$ (3)

$(K + 68.0398) > 0 \ \rightarrow K > -68.0398$ (4)

The 2nd Scenario is accepted

from row S^1: $-22.8K^3 + 5624.304K^2 - 334003.584K + 3930143.616 > 0$

$$K^3 - 246.68K^2 + 14649.28K - 172374.72 < 0$$

$$(K - 163.5568)(K - 67.5126)(K - 15.6106) < 0$$

In that case we have 3 scenarios:

First Scenario

$(K - 163.5568) > 0 \ \rightarrow K > 163.5568$

$(K - 67.5126) > 0 \ \rightarrow K > 67.5126$

$(K - 15.6106) < 0 \ \rightarrow K < 15.6106$

Second Scenario

$(K - 163.5568) > 0 \ \rightarrow K > 163.5568$

$(K - 67.5126) < 0 \ \rightarrow K < 67.5126$

$(K - 15.6106) > 0 \ \rightarrow K > 15.6106$

Third Scenario:

$$(K - 163.5568) < 0 \ \rightarrow K < 163.5568 \quad (5)$$
$$(K - 67.5126) > 0 \ \rightarrow K > 67.5126 \quad (6)$$
$$(K - 15.6106) > 0 \ \rightarrow K > 15.6106$$

The range of K for stability is $0 < K < 15.6106$ & $67.5126 < K < 163.5568$##

b) The maximum value of K for stability is at K = 15.6106 & K=67.5126 & K=163.5568

This value is obtained from the row S^1 so the auxiliary equation A(S) is obtained from S^2

$$A(s) = (-K^2 + 143.08K + 14364.56)S^2 + 4K(401 - K) = 0$$

At K= 15.6106;

$16354.43382\ S^2 + 24064.63907 = 0 \quad \rightarrow S = \pm J1.213$ rad/s

At K= 67.5126;

$19466.31165\ S^2 + 90058.40576 = 0 \quad \rightarrow S = \pm J2.1509$ rad/s

At K= 163.5568;

$11015.44012\ S^2 + 155341.8 = 0 \quad \rightarrow S = \pm J3.7553$ rad/s

Sheet 7 *(Stability of linear systems)*

Problem #1

Utilizing the Routh-Hurwitz criterion, determine the stability of the following polynomials:

(a) $q(s) = s^2 + 5s + 2$

(b) $q(s) = s^3 + 20s^2 + 5s + 100$

(c) $q(s) = s^3 + 3s^2 + 4s + 2$

(d) $q(s) = s^3 + 2s^2 - 4s + 20$

(e) $q(s) = s^4 + s^3 + 2s^2 + 10s + 8$

(f) $q(s) = s^5 + s^4 + 2s^3 + s + 5$

(g) $q(s) = s^5 + s^4 + 2s^3 + s^2 + s + 15$

(h) $q(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16$

Problem #2

Utilizing the Routh-Hurwitz criterion, determine the range of K that results in a stable system of the following characteristic equations:

(a) $q(s) = s^3 + 10s^2 + 29s + K$

(b) $q(s) = s^3 + 3s^2 + (K + 1)s + 6$

(c) $q(s) = s^3 + (K + 2)s^2 + 2Ks + 10$

(d) $q(s) = s^4 + s^3 + 3s^2 + 2s + K$

(e) $q(s) = s^4 + 2s^3 + (4 + K)\, s^2 + 9s + 25$

(f) $q(s) = s^4 + Ks^3 + 5s^2 + 10s + 10K$

(g) $q(s) = s^4 + Ks^3 + 2s^2 + (K + 1)s + 10$

(h) $q(s) = s^5 + s^4 + 2s^3 + s^2 + s + K$

Problem #3

A feedback control system has a characteristic equation

$$q(s) = s^3 + (1 + K)s^2 + 10s + (5 + 15K)$$

The parameter K must be positive. What is the maximum value K can assume before the system becomes unstable? When K is equal to the maximum value, the system oscillates. Determine the frequency of oscillation.

Problem #4

Consider the closed loop system given in Fig. 1. Find the range of values of *K* for which the system is stable.

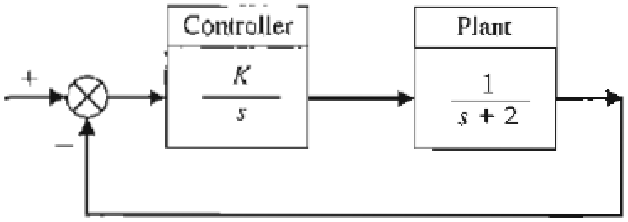


Fig. 1, A closed-loop control system

Problem 5

Designers have developed small, fast, vertical-takeoff fighter aircraft that are invisible to radar (stealth aircraft). This aircraft concept uses quickly turning jet nozzles to steer the airplane. The control system for the heading or direction control is shown in Fig. 2. Determine the maximum gain of the system for stable operation.

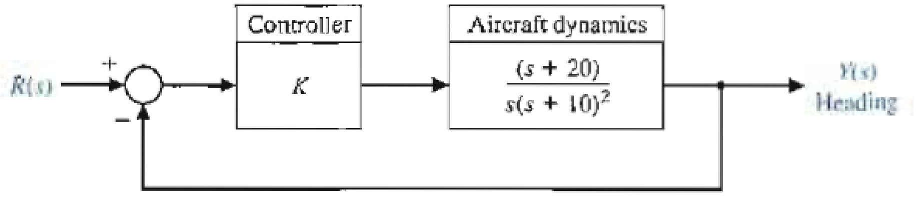


Fig. 2, Aircraft heading control

Problem 6

Consider the system given in Fig. 3. Find the range of values of *K* for which the system is stable.

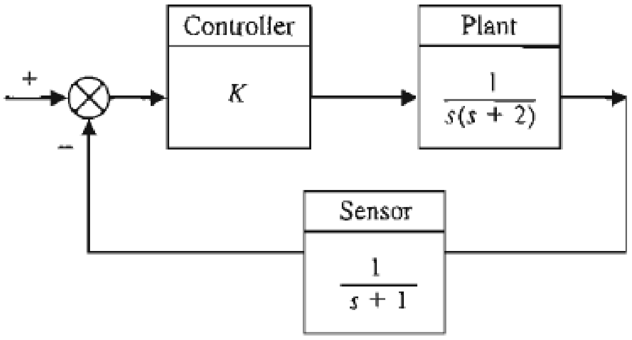


Fig. 3, A closed-loop system

Problem 7

A closed-loop feedback system is shown in Fig. 4. For what range of values of the parameters *K* and *p* is the system stable?

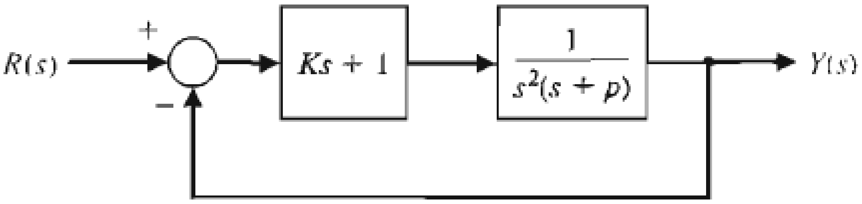


Fig. 4, Closed-loop system with parameters *K* and *p*

Problem 8

Arc welding is one of the most important areas of application for industrial robots. In most manufacturing welding situations, uncertainties in dimensions of the part, geometry of the joint, and the welding process itself require the use of sensors for maintaining weld quality. Several systems use a vision system to measure the geometry of the puddle of melted metal, as shown in Fig. 5. This system uses a constant rate of feeding the wire to be melted.

Calculate the maximum value for *K* for the system that will result in a stable system.

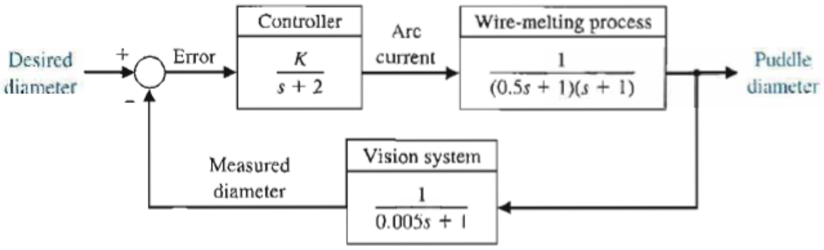


Fig. 5, Welder control

Problem 9

A cassette tape storage device has been designed for mass-storage. It is necessary to control the velocity of the tape accurately. The speed control of the tape drive is represented by the system shown in Fig. 6. Determine the limiting gain for a stable system.

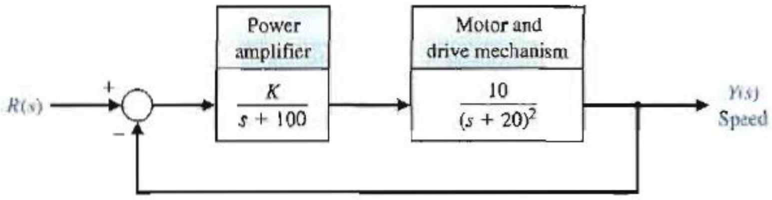


Fig. 6, Tape drive control

Problem 10

Robots can be used in manufacturing and assembly operations that require accurate, fast, and versatile manipulation. The open-loop transfer function of a direct-drive arm may be approximated by

$$G(s)H(s)=\frac{K(s+10)}{s(s+3)(s^2+4s+8)}$$

(a) Determine the value of gain K when the system oscillates,
(b) Calculate the roots of the closed-loop system for the K determined in part (a).

Problem 11

Given the forward –path transfer function of a unity feedback control systems,

$$G(s)=\frac{K(s+4)(s+20)}{s^3(s+100)(s+500)} \qquad G(s)=\frac{K(s+10)(s+20)}{s^2(s+2)}$$

$$G(s)=\frac{K}{s(s+10)(s+20)} \qquad G(s)=\frac{K(s+1)}{(s^2+2s^2+3s+1)}$$

(a) Apply the Routh-Hurwitz criterion to determine the stability of the closed–loop system as function of K.

(b) Determine the values of K that will cause sustained constant amplitude oscillations in the system.

(c) Determine the frequency of oscillation.

Problem 12

Consider the following Routh table. Notice that the s^5 row was originally all zeros. Tell how many roots of the original polynomial were in the right-half plane, in the left-half plane, and on the jω-axis.

s^7	1	2	−1	−2
s^6	1	2	−1	−2
s^5	3	4	−1	
s^4	1	−1	−3	
s^3	7	8		
s^2	−15	−21		
s^1	−9			
s^0	−21			

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