SPC318: System Modeling and Linear Systems

Lecture 2: Mathematical Modeling of Mechanical and Electrical Systems

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Lecture Outline:

1. Remarks on The System Transfer Function.
2. Linearization of Non-linear Systems.
3. Mathematical Modeling of Mechanical Systems.
4. Mathematical Modeling of Electrical Systems.
5. Mathematical Modeling of Electromechanical Systems.

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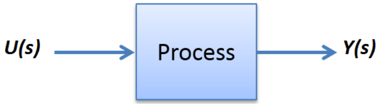
Remarks on The System Transfer Function:

Transfer function of Linear Systems:



$$G(s) = \frac{numerator}{denominator} = \frac{\mathcal{L}[output]}{\mathcal{L}[input]} = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} = \frac{p(s)}{q(s)} \qquad (n \geq m)$$

Remarks:

1. If the **highest power** of s in the **denominator** of the transfer function is equal to n, the system is called an n^{th}-**order** system. (e.g. $G(s) = \frac{s+1}{s^2+2s-1}$ is a second-order system)
2. When order of the denominator polynomial is greater than the numerator polynomial the transfer function is said to be "proper". Otherwise "improper".
3. "Improper" transfer function could not exist **physically**.

Remarks on The System Transfer Function:

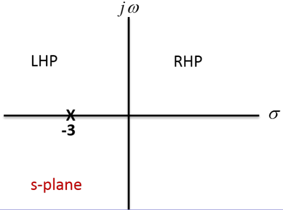
Transfer function of Linear Systems:

$$G(s) = \frac{numerator}{denominator} = \frac{\mathcal{L}[output]}{\mathcal{L}[input]} = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = \frac{p(s)}{q(s)} \qquad (n \geq m)$$

Poles and Zeros:

1. Roots of denominator polynomial, $q(s) = 0$, are called '**poles**'.
2. Roots of numerator polynomial, $p(s) = 0$, are called '**zeros**'.
3. Poles are represented by x on s-plane.
4. Zeros are represented by o on s-plane.

$$G(s) = \frac{10}{s+3}$$



Remarks on The System Transfer Function:

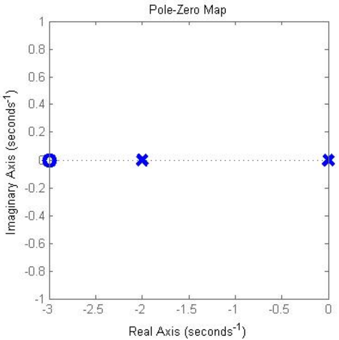
Transfer function of Linear Systems:

Consider the following transfer functions:

1. Determine whether the transfer function is proper or improper.
2. Calculate the Poles and zeros of the system.
3. Determine the order of the system.
4. Draw the pole-zero map.

- $$G(s) = \frac{s+3}{s(s+2)}$$
- $$G(s) = \frac{(s+3)^2}{s(s^2+10)}$$

```
%% (1) Enter the system in transfer function:
sys = tf([1 3],[1 2 0]); % G(s) = (s+3)/s(s+2)
%% (2) Find the system order:
sys_or = order(sys); %
%% (3) Put the system in zero-pole-gain format:
zpk(sys);
%% (4) Find zeros and poles of the system:
z=zero(sys);
p=pole(sys);
%% (5) Draw the poles and zeros on the s-plane:
pzplot(sys);
```



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Linearization of Non-linear Systems:

Non-linear system

A system is nonlinear if the principle of **superposition** and **homogeneous** are not applied.

- In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve **nonlinear relationships** among the variables.
- The non-linear systems are assumed to behave as **linear** system for a **limited operating range**.
- Example of nonlinear system is the damping force. It is linear at low velocity operation and non-linear at high velocity operation.

Linearization of Nonlinear Systems:

If the system operates around an **equilibrium point** and if the signals involved are **small signals**, then it is possible to approximate the nonlinear system by a linear system.

Linearizion of Non-linear Systems:

Linear Approximation of Nonlinear Mathematical Models:

Consider a non-linear system defined by:

$$y = f(u) \qquad (1)$$

To obtain a linear model we assume that the variables deviate **slightly** from some operating condition corresponds to $\bar{u}$ and $\bar{y}$. The equation (1) can be expanded by using Taylor expansion:

$$y = f(u)$$

$$y = f(\bar{u}) + \dot{f}(\bar{u})(u - \bar{u}) + \frac{1}{2!}\ddot{f}(\bar{u})(u - \bar{u})^2 + \dots$$

If the deviation $(u - \bar{u})$ is small, we can neglect the high derivative terms:

$$\boxed{y = f(\bar{u}) + \dot{f}(\bar{u})(u - \bar{u})}$$

Linearizion of Non-linear Systems:

Linear Approximation of Nonlinear Mathematical Models:

If the system is non-linear and has two inputs u_1 and u_2:

$$y = f(u_1, u_2)$$

The linearized model could be obtained by:

$$\boxed{y = f(\bar{u_1}, \bar{u_2}) + \frac{\partial f(\bar{u_1})}{\partial u_1}(u_1 - \bar{u_1}) + \frac{\partial f(\bar{u_2})}{\partial u_2}(u_2 - \bar{u_2})}$$

Example: Linearize the system:

$$z = xy$$

in the region $5 \leq x \leq 7$, $10 \leq y \leq 12$.

Solution: Choose $\bar{x} = 6$ and $\bar{y} = 11$

$$f(\bar{x}, \bar{y}) = 66; \quad \frac{\partial f(\bar{x})}{\partial x} = 11; \quad \frac{\partial f(\bar{y})}{\partial y} = 6$$

The linearized model is

$$\boxed{z = 6(x) + 11(y) - 66}$$

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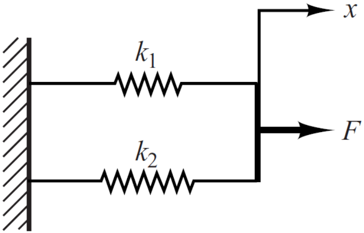
Mathematical Modeling of Mechanical Systems:

Equivalent Spring Constant:

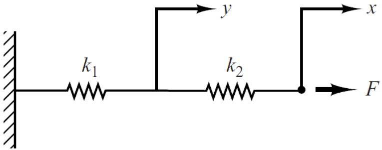
Connected in Parallel



$$F = k_1 x + k_2 x = k_{eq} x$$

$$\boxed{k_{eq} = k_1 + k_2}$$

Connected in Series



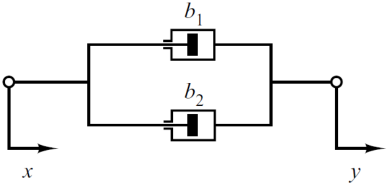
$$F = k_1 y = k_2 (x - y)$$

$$\boxed{k_{eq} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}} = \frac{k_1 k_2}{k_2 + k_2}}$$

Mathematical Modeling of Mechanical Systems:

Equivalent Friction Constant:

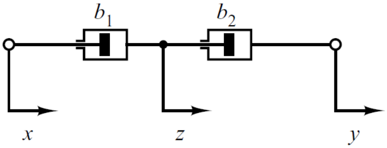
Connected in Parallel



$$F = b_1(\dot{z} - \dot{x}) + b_2(\dot{y} - \dot{x})$$

$$\boxed{b_{eq} = b_1 + b_2}$$

Connected in Series



$$F = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{x})$$

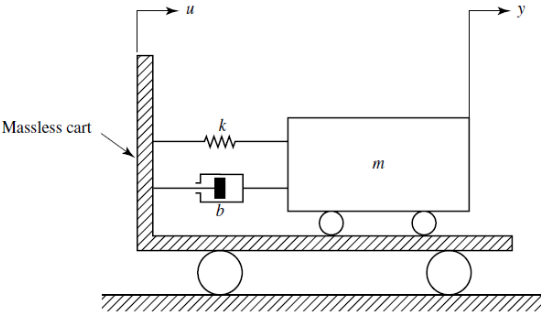
$$\boxed{b_{eq} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}} = \frac{b_1 b_2}{b_2 + b_2}}$$

Mathematical Modeling of Mechanical Systems:

Example 1:

Spring-mass-damper system mounted on a cart

Consider the spring-mass-damper system mounted on a massless cart, $u(t)$ is the displacement of the cart and is the input to the system. The displacement $y(t)$ of the mass is the output. In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant.



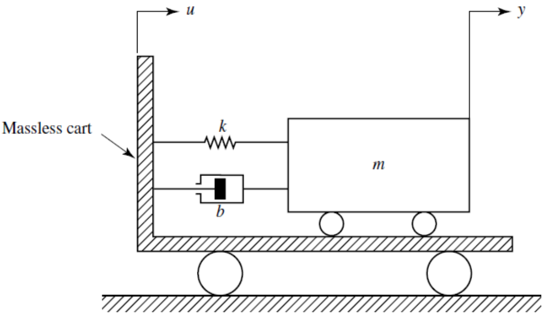
For **translational** systems, Newton's second law is used:

$$\boxed{ma = \sum F}$$

m is the mass.
a is the acceleration.
F is the force.

Mathematical Modeling of Mechanical Systems:

Example 1:



$$\boxed{ma = \sum F}$$

$$m\frac{d^2y}{dt^2} = -b(\frac{dy}{dt} - \frac{du}{dt}) - k(y - u)$$

Taking the Laplace transform of this last equation, assuming zero initial condition:

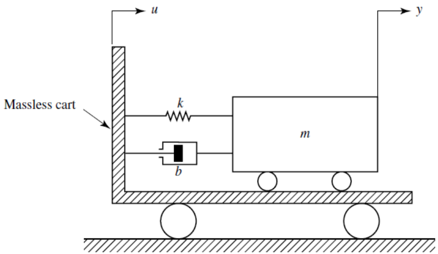
$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

The transfer function:

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}}$$

Mathematical Modeling of Mechanical Systems:

Example 1:



To obtain a state-space model of this system:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

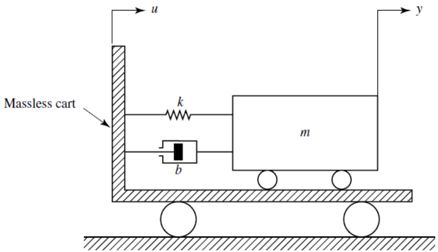
1. Write the system differential equation.

$$m\frac{d^2y}{dt^2} = -b(\frac{dy}{dt} - \frac{du}{dt}) - k(y - u)$$

$$m\ddot{y} = -b\dot{y} - ky + b\dot{u} + ku$$

Mathematical Modeling of Mechanical Systems:

Example 1:



To obtain a state-space model of this system:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

2. Put the **output highest derivative** at one side:

$$\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{b}{m}\dot{u} + \frac{k}{m}u$$

Mathematical Modeling of Mechanical Systems:

Example 1:

$$\ddot{y} = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \frac{b}{m}\dot{u} + \frac{k}{m}u$$

3. Define two states:

$$\boxed{x_1 = y}$$

$$x_2 = \dot{y} - \frac{b}{m}u \qquad Why?$$

4. Differentiate the two states:

$$\boxed{\dot{x}_1 = \dot{y} = x_2 + \frac{b}{m}u}$$

$$\dot{x}_2 = \ddot{y} - \frac{b}{m}\dot{u}$$

$$\dot{x}_2 = -\frac{b}{m}\dot{y} - \frac{k}{m}y + \cancel{\frac{b}{m}\dot{u}} + \frac{k}{m}u - \cancel{\frac{b}{m}\dot{u}}$$

$$\dot{x}_2 = -\frac{b}{m}[x_2 + \frac{b}{m}u] - \frac{k}{m}[x_1] + \frac{k}{m}u$$

$$\boxed{\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + ((\frac{b}{m})^2 + \frac{k}{m})u}$$

Mathematical Modeling of Mechanical Systems:
Example 1:

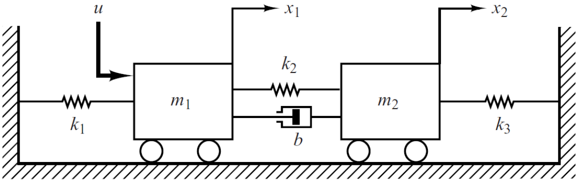
5. Write the equations in state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{b}{m} \\ \dfrac{k}{m} - (\dfrac{b}{m})^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Mathematical Modeling of Mechanical Systems:

Example 2:



(1) Equation of motion:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

(2) Simplifying,

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

(3) Laplace transform,

$$[m_1s^2 + bs + (k_1 + k_2)]X_1(s) = (bs + k_2)X_2(s) + U(s)$$

$$[m_2s^2 + bs + (k_2 + k_3)]X_2(s) = (bs + k_2)X_1(s)$$

(4) Substitute by $X_2(s)$,

$$[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]X_1(s) = (m_2s^2 + bs + k_2 + k_3)U(s)$$

(5) Finally,

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2}$$

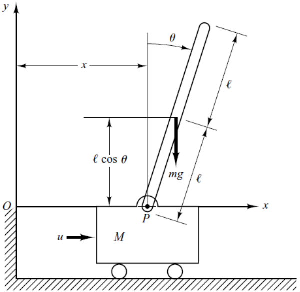
$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2}$$

Mathematical Modeling of Mechanical Systems:

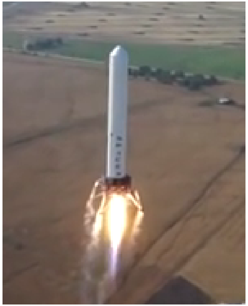
Example 3:

Inverted Pendulum

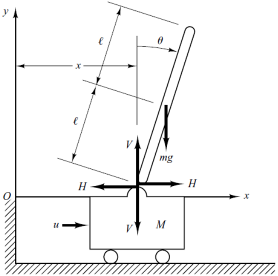
An inverted pendulum mounted on a motor-driven cart. The inverted pendulum is naturally unstable in that it may fall over any time in any direction unless a suitable control force is applied.



Inverted Pendulum



Solid Rocket Booster



Free-body diagram

Mathematical Modeling of Mechanical Systems:

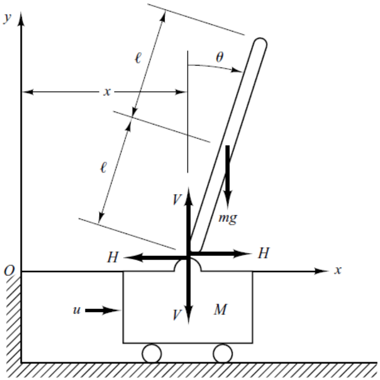
Example 3:

- Define u as the input force.
- The rotational motion of the pendulum rod around its center of gravity:

$$\boxed{I\ddot{\theta} = \sum Moments}$$

$$I\ddot{\theta} = V * L * sin\theta - H * L * cos\theta$$

I : Mass moment of inertia. $(kg.m^2)$
θ: Rotational angle.
V: Vertical reaction force.
H: Horizontal reaction force.
L: Half length of the rod.



Free-body diagram

Mathematical Modeling of Mechanical Systems:

Example 3:

- The horizontal motion of rod center of gravity:

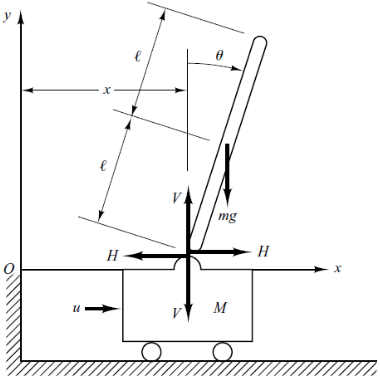
$$\boxed{ma = \sum F}$$

$$m\frac{d^2}{dt^2}(x + L * sin\theta) = H$$

- The vertical motion of rod center of gravity:

$$\boxed{ma = \sum F}$$

$$m\frac{d^2}{dt^2}(L * cos\theta) = V - mg$$



Free-body diagram

Mathematical Modeling of Mechanical Systems:

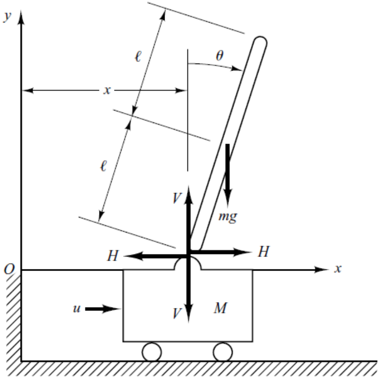
Example 3:

- The horizontal motion of the cart:

$$\boxed{Ma = \sum F}$$

$$M\frac{d^2x}{dt^2} = u - H$$

- Since we need to keep the pendulum vertical, we can assume θ and $\dot{theta}$ are small quantities. So,
  - $sin\theta \approx \theta$.
  - $cos\theta = 1$.
  - $\theta\dot{\theta}^2 = 0$.



Free-body diagram

Mathematical Modeling of Mechanical Systems:

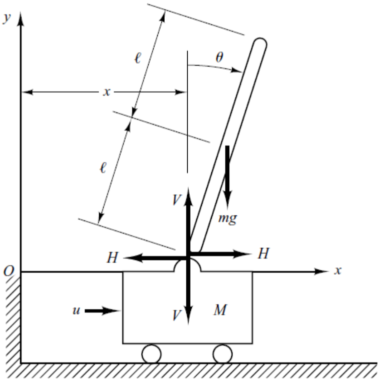
Example 3:

- Using the linearity assumptions:

1.
$$I\ddot{\theta} = V * L * sin\theta - H * L * cos\theta$$
$$\boxed{I\ddot{\theta} = V * L * \theta - H * L} \qquad (1)$$

2.
$$m\frac{d^2}{dt^2}(x + L * sin\theta) = H$$
$$\boxed{m(\ddot{x} + L\ddot{\theta}) = H} \qquad (2)$$

3.
$$m\frac{d^2}{dt^2}(L * cos\theta) = V - mg$$
$$\boxed{0 = V - mg} \qquad (3)$$



Free-body diagram

Mathematical Modeling of Mechanical Systems:

Example 3:

- From the cart horizontal motion:

$$H = u - M\ddot{x}$$

So, substitute by H in (2):

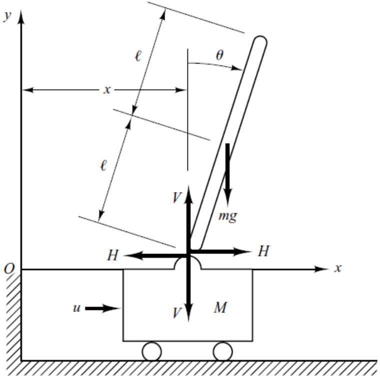
$$(M + m)\ddot{x} + m * L * \ddot{\theta} = u$$

- From the pendulum equations (1),(2) and (3):

$$V = mg$$

So,

$$(I + mL^2)\ddot{\theta} + m * L * \ddot{x} = m * g * L * \theta$$



Free-body diagram

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Mathematical Modeling of Electrical Systems:

Electrical Resistance, Inductance and Capacitance:

Resistance

V-I in time domain

$$\nu_R(t) = i_R(t)R$$

V-I in s domain

$$V_R(s) = I_R(s)R$$

Inductance

V-I in time domain

$$\nu_L(t) = L\frac{di_L(t)}{dt}$$

V-I in s domain

$$V_L(s) = sLI_L(s)$$

Capacitance

V-I in time domain

$$\nu_c(t) = \frac{1}{C}\int i_c(t)dt$$

V-I in s domain

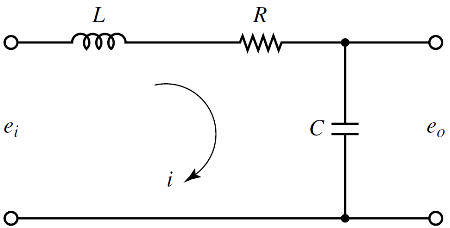
$$V_c(s) = \frac{1}{Cs}I_c(s)$$

Mathematical Modeling of Electrical Systems:

Example 1:

RLC circuit

We need to find the transfer function $G(s) = \frac{E_o(s)}{E_i(s)}$ of the RLC network.



RLC circuit

Applying the **Kirchhoff's voltage law:**

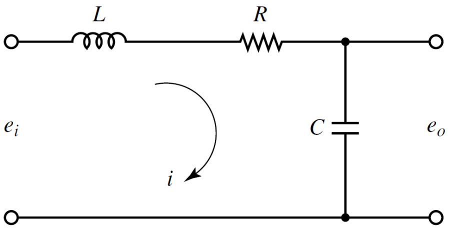
$$\boxed{\sum V = 0}$$

$$e_i(t) - L\frac{di}{dt} - R.i - \frac{1}{C}\int i \; dt = 0$$

$$\frac{1}{C}\int i \; dt = e_o$$

Mathematical Modeling of Electrical Systems:

Example 1:



RLC circuit

Taking Laplace transform with zero initial conditions:

$$L.s.I(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

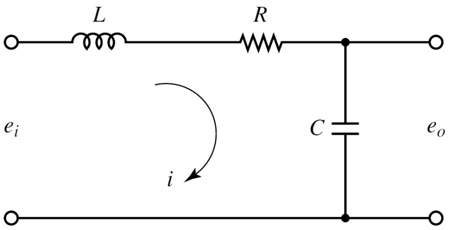
$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

So,

$$\boxed{G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Mathematical Modeling of Electrical Systems:

Example 1:



RLC circuit

To find the state-space model from TF:

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

The differential equation for the system:

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Defining state variables:

$$x_1 = e_o = y$$

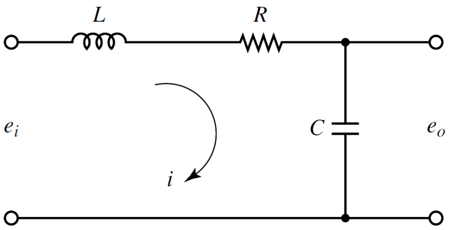
$$\dot{x}_1 = \dot{e}_o = x_2$$

$$x_2 = \dot{e}_o$$

$$\dot{x}_2 = \ddot{e}_o = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{LC}u$$

Mathematical Modeling of Electrical Systems:

Example 1:



RLC circuit

Put equations in state-space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

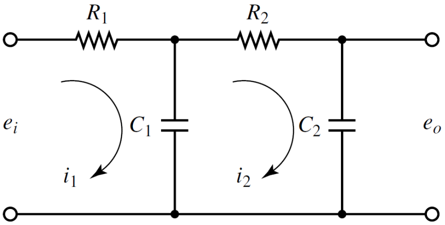
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Mathematical Modeling of Electrical Systems:

Example 2:

Cascaded RC circuit

We need to find the transfer function $G(s) = \dfrac{E_o(s)}{E_i(s)}$ of the cascaded RC network.



RLC circuit

Applying the **Kirchhoff's voltage law:**

$$\boxed{\sum V = 0}$$

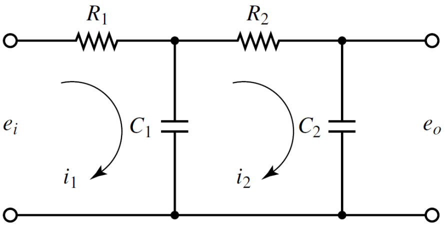
$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0$$

$$\frac{1}{C_2}\int i_2\,dt = e_o$$

Mathematical Modeling of Electrical Systems:

Example 2:



RLC circuit

Taking Laplace transform:

$$\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{C_1 s}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s)$$

Eliminate $I_1(s)$ and $I_2(s)$. So,

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s}$$

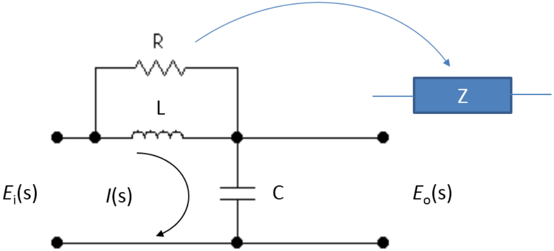
$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

Mathematical Modeling of Electrical Systems:

Example 3:

Series/Parallel RLC

We need to find the transfer function $G(s) = \dfrac{E_o(s)}{E_i(s)}$ of the cascaded RC network.



RLC circuit

Series/Parallel RLC

We need to find the equivalent impedance Z for the connected components.

Mathematical Modeling of Electrical Systems:

Equivelent Impedance:

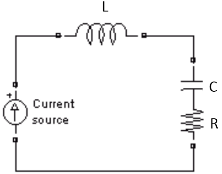
$$Z_R(s) = R$$

$$Z_L(s) = Ls$$

$$Z_c(s) = \frac{1}{Cs}$$

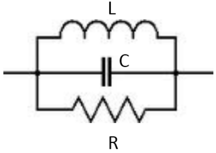
Series/Parallel Impedance

Series



$$Z_T = Z_1 + Z_2 + Z_3$$

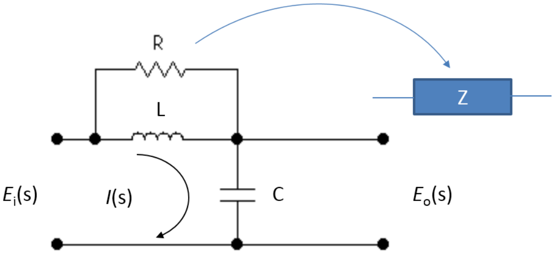
Parallel



$$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

Mathematical Modeling of Electrical Systems:

Example 3:



RLC circuit

Equivalent Impedance of R and L:

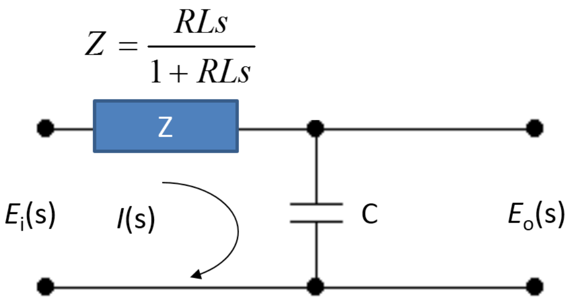
$$\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

$$\frac{1}{Z_T} = \frac{1}{R} + \frac{1}{Ls}$$

$$Z_T = \frac{RLs}{1 + RLs}$$

Mathematical Modeling of Electrical Systems:

Example 3:



RLC circuit

$$\sum V = 0$$

$$E_i(s) = I(s)Z_T + \frac{1}{Cs}I(s) \qquad (1)$$

$$E_o(s) = \frac{1}{Cs}I(s) \qquad (2)$$

Divide (2) by (1) to find $G(s) = \frac{E_o(s)}{E_i(s)}$

Table of Contents

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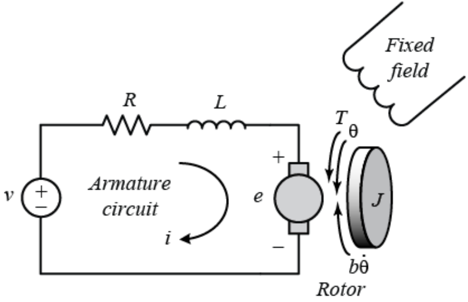
DC Motor

An actuator, converting **electrical** energy into rotational **mechanical** energy. For this example, the **input** of the system is the **voltage source** (ν) applied to the motor's armature, while the output is the rotational speed of the shaft $\dot{\theta}$.



DC Motor

- For the armature electrical circuit KVL:

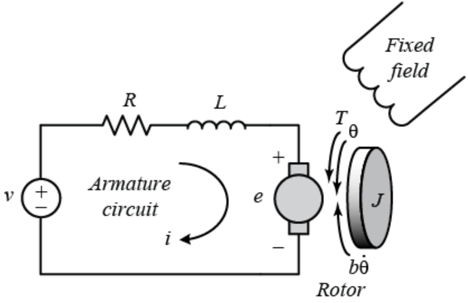
$$V - V_{emf} - L\frac{di}{dt} - Ri = 0$$

The back emf, V_{emf}, is proportional to the angular velocity of the shaft, $\dot{\theta}$, by a constant factor K_e. So,

$$\boxed{V - K_e\dot{\theta} - L\frac{di}{dt} - Ri = 0}$$

Mathematical Modeling of Electromechanical Systems:

Mathematical Modeling DC Motor:



DC Motor

- For the shaft mechanical system:

$$J\ddot{\theta} = T_{motor} - b\dot{\theta}$$

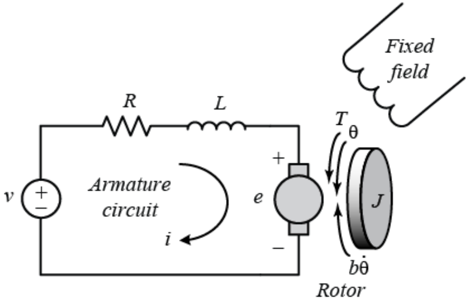
  $b\dot{\theta}$ is the viscous damping force.
  The motor torque T_{motor} is proportional to the armature current i by a constant factor K_t. So,

$$\boxed{J\ddot{\theta} = K_t i - b\dot{\theta}}$$

- in SI units, the K_t and constants are equal, that is, $K_t = K_e = K$.

Mathematical Modeling of Electromechanical Systems:

Mathematical Modeling DC Motor:



DC Motor

- By taking the Laplace transform,

$$\boxed{V - K\dot{\theta} - L\frac{di}{dt} - Ri = 0}$$

$$V(s) = Ks\theta(s) + Ls * I(s) + RI(s) \qquad (1)$$

$$\boxed{J\ddot{\theta} = K * i - b\dot{\theta}}$$

$$Js^2\theta(s) = KI(s) - b\theta(s) \qquad (2)$$

Eliminate $I(s)$ chose $s\theta(s) = W(s)$ as the rotational speed:

$$G(s) = \frac{W(s)}{V(s)} = \frac{K}{(Js + b)(Ls + R) + K^2}$$

End of Lecture

Best Wishes

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