SPC318: System Modeling and Linear Systems

Lecture 7: Closed Loop Control Systems
Final Value Theorem and Steady State Error

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Lecture Outline:

1. The Final Value Theorem.
2. The Steady State Error.

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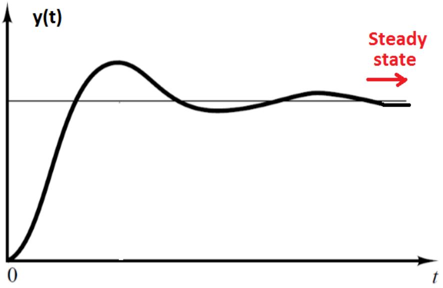
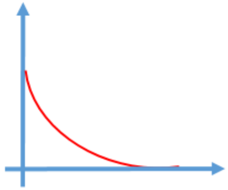
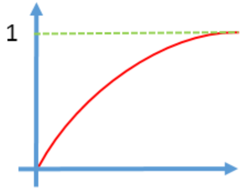
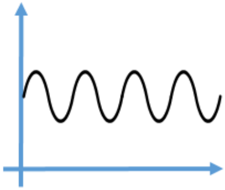
The Final Value Theorem:

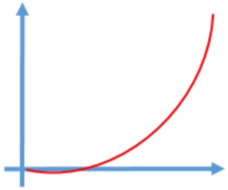
- Sometimes, it is important to know the **final value** (FV) of the system response or the **steady state value**, $y(\infty)$.
- Finding the $y(\infty)$ is impossible mathematically. So, we can find the final value by computing the limits when time approaches infinity. ($\lim_{t\to\infty} y(t)$).











The Final Value Theorem:

- If the output **converges** to a single value, the final value is exist.
- If the output **diverges** to infinity or **oscillates** continuously, no meaning for the final value.

How do we calculate final value ?

Time Domain:

$$\text{Final Value} = f(t=\infty) = \lim_{t\to\infty} y(t)$$

- If we have a system defined by differential equations, it requires a lot of calculations to find the final value in time-domain.
- We need to find the final value in S-domain directly.

Example

$$\ddot{y} + 3\dot{y} + 2y = \delta(0) \quad \text{what} \lim_{t\to\infty} ?$$

in S-domain:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1 \qquad \text{Algebric}$$

What is the S-domain version of $\lim_{t\to\infty} y(t)$?

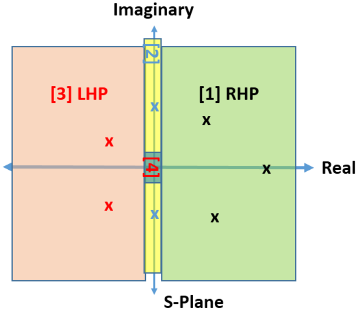
The Final Value Theorem:

Final Value Theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

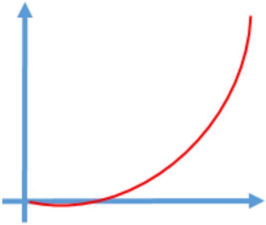
1. Find Laplace transform, $Y(s)$.
2. Multiply it by s.
3. Find $\lim_{s \to 0}$.

Advantage: Easy to calculate since most of the time we have the system represented in s-domain.
But: We can not always use the final value theorem !



The Final Value Theorem:

RHP

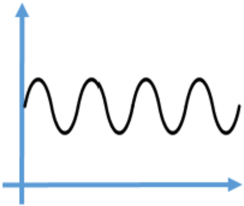


$$Y(s) = \frac{1}{s-2}$$

$$FVT = \lim_{s\to 0} sY(s) = 0$$

Wrong!

jw-axis

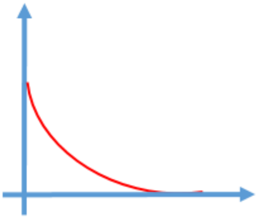


$$Y(s) = \frac{1}{s^2+4}$$

$$FVT = \lim_{s\to 0} sY(s) = 0$$

Wrong!

LHP

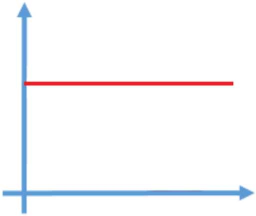


$$Y(s) = \frac{1}{s+2}$$

$$FVT = \lim_{s\to 0} sY(s) = 0$$

(Correct!)

Origin (integrator)



$$Y(s) = \frac{1}{s}$$

$$FVT = \lim_{s\to 0} sY(s) = 1$$

(Correct!)

Don't use the Final Value Theorem if the system poles lie in the RHP or on the jw-axis!

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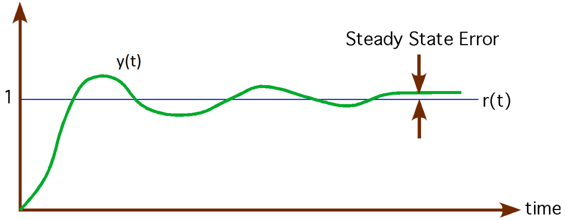
The Steady State Error:

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.
- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.
- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.
- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.



The Steady State Error:

Definition (System Type)

- Consider the unity-feedback control system with the following **open-loop** transfer function:

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

- s^N in the denominator, represents N poles at the origin.
- A system is called type 0, type 1, type 2, ... , if $N = 0$, $N = 1$, $N = 2$, ... , respectively.

Examples

$$G(s) = \frac{s+1}{s} \qquad \text{type 1}$$

$$G(s) = \frac{s-3}{s^2} \qquad \text{type 2}$$

$$G(s) = \frac{s+1}{s+2} \qquad \text{type 0}$$

The Steady State Error:

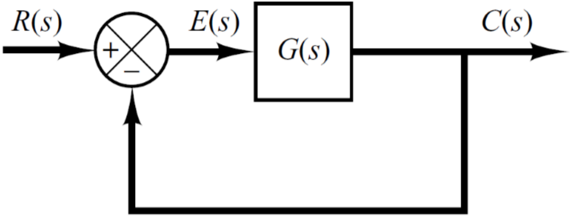
- The transfer function between the error signal $E(s)$ and the input signal $R(s)$ is:

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

- The final-value theorem provides a convenient way to find the **steady-state performance** of a stable system.

- $$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$\boxed{e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}}$$



The Steady State Error:

[1] Unit Step Input:

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s}$$

$$\boxed{e_{ss} = \frac{1}{1 + G(0)}}$$

K_p: Static position error:

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

$$e_{ss} = \frac{1}{1 + K_p}$$

For type 0 system:

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = K$$

$$e_{ss} = \frac{1}{1 + K}$$

For type 1 system:

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{s^N (T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \qquad \text{for } N \geq 1$$

$$e_{ss} = 0$$

The Steady State Error:

[2] Unit Ramp Input:

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$\boxed{e_{ss} = \frac{1}{sG(s)}}$$

K_v: Static velocity error:

$$K_v = \lim_{s \to 0} sG(s)$$

$$e_{ss} = \frac{1}{K_v}$$

For type 0 system:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

$$e_{ss} = \frac{1}{K_v} = \infty$$

For type 1 system:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}$$

For type 2 system:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

$$e_{ss} = \frac{1}{K_v} = 0$$

The Steady State Error:

[3] Unit Parabolic Input:

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^3}$$

$$\boxed{e_{ss} = \frac{1}{s^2 G(s)}}$$

K_a: Static acceleration error:

$$K_a = \lim_{s\to 0} s^2 G(s)$$

$$e_{ss} = \frac{1}{K_a}$$

For type 0 system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

$$e_{ss} = \frac{1}{K_a} = \infty$$

For type 1 system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

$$e_{ss} = \frac{1}{K_a} = \infty$$

For type 2 system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{K}$$

For type 3 system:

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \quad \text{for } N \geq 3$$

$$e_{ss} = \frac{1}{K_a} = 0$$

The Steady State Error:

Summary:

Input Type

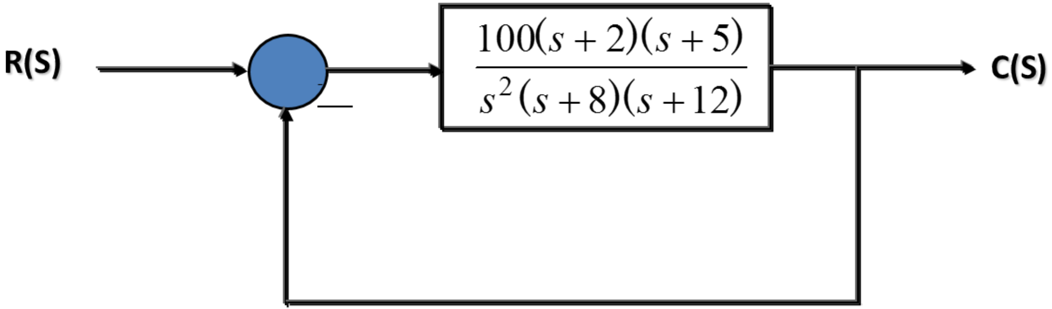
System Type

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = t^2$
Type 0 system	$\frac{1}{1 + K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Steady state error

Example

For the system shown in figure below evaluate the **static error constants** K_p, K_v and K_a and find the **expected steady state errors** for the unit step, ramp and parabolic inputs.



Solution

$$G(s)=\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p=\lim_{s\to 0}G(s)$$

$$K_p=\lim_{s\to 0}\left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_p=\infty$$

$$K_v=\lim_{s\to 0}sG(s)$$

$$K_v=\lim_{s\to 0}\left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_v=\infty$$

$$K_a=\lim_{s\to 0}s^2G(s)$$

$$K_a=\lim_{s\to 0}\left(\frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_a=\left(\frac{100(0+2)(0+5)}{(0+8)(0+12)}\right)=10.4$$

$$e_{ss}=\frac{1}{1+K_p}=0$$ **Unit-step**

$$e_{ss}=\frac{1}{K_v}=0$$ **Ramp**

$$e_{ss}=\frac{1}{K_a}=0.09$$ **Parabolic**

QUIZ!

By using Routh table, determine K values that guarantee stability for the following characteristic equation:

$$s^4 + 10s^3 + 35s^2 + (50 + K)s + (24 - 2K) = 0$$

Time Allowed: **10** Minutes.

End of Lecture

Best Wishes

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