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Soliton Solutions of Nonlinear Evolutions Equation by using the Extended $\exp(-\varphi(\xi))$ Expansion Method

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ABSTRACT

In this research, we employ the extended $\exp(-\varphi(\xi))$ expansion method for the first time to obtain the exact and solitary wave solutions of the (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama Equation. We obtain the wide range of exact and solitary wave solutions of distinct physical structure.

Keywords

Extended $\exp(-\varphi(\xi))$ -expansion method, The (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama equation, Traveling wave solutions, Solitary wave solutions.

1. INTRODUCTION

Most of complicated phenomena in physics and different branches of applied science (plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena...etc.) can be represented by nonlinear partial differential equations (NLPDEs.) which are widely used to describe them. The nonlinear partial differential equations of mathematical physics are major subjects in physical science [1]. Investigation the exact solutions of NLPDEs. help us to understand the nonlinear physical phenomena. Recently, many solutions methods have been introduced to get the exact solutions of NLPDEs. For example, extended Jacobian Elliptic Function Expansion Method [2], the modified simple equation method[3], the tanh method [4], extended tanh - method [5]-[7], sine - cosine method [8]-[10], homogeneous balance method [11, 12], F-expansion method [13]-[15], exp-function method [16, 17], trigonometric function series method [18], $(\frac{G'}{G})$ -expansion method [19]-[22], Jacobi elliptic function method [23]-[26], The $\exp(-\varphi(\xi))$ -expansion method[27]-[29] and so on. In this article we propose a new method to get the exact traveling wave solutions and the solitary wave solutions of the (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama Equation which is a widely used model for investigation the dynamics of solitons and nonlinear wave in areas such as fluid dynamics, plasma physics and weakly dispersive media by using the extended $\exp(-\varphi(\xi))$ -expansion method.

The rest of this paper is organized as follows: In Section 2, we give the description of The extended $\exp(-\varphi(\xi))$ -expansion method In

Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. DESCRIPTION OF METHOD

Consider the following nonlinear evolution equation in the polynomial form

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \quad (1)$$

in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method

Step 1. We use the wave transformation

$$u(x, y, z, t) = u(\xi), \quad \xi = x + y + z - \omega t, \quad (2)$$

where c is a positive constant, to reduce Eq. (1) to the following ODE:

$$P(u, u', u'', u''', \dots) = 0, \quad (3)$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$.

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $\exp(-\varphi(\xi))$ as follows

$$u(\xi) = \sum_{i=-m}^m a_i (\exp(-\varphi(\xi)))^i, \quad (4)$$

Since a_m ($0 \leq m \leq n$) are constants to be determined, such that $a_m \text{ or } a_{-m} \neq 0$.

the positive integer m can be determined by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in Eq. (3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = m$, which gives rise to degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, \quad D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n + q).$$

Therefore, we can find the value of m in Eq. (3), where $\varphi = \varphi(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda, \quad (5)$$

the solutions of ODE (3) are
when $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$\varphi(\xi) = \ln \left(\frac{-\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right), \quad (6)$$

and

$$\varphi(\xi) = \ln \left(\frac{-\sqrt{\lambda^2 - 4\mu} \coth \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right), \quad (7)$$

when $\lambda^2 - 4\mu > 0, \mu = 0$,

$$\varphi(\xi) = -\ln \left(\frac{\lambda}{\exp(\lambda(\xi + C_1)) - 1} \right), \quad (8)$$

when $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$\varphi(\xi) = \ln \left(-\frac{2(\lambda(\xi + C_1) + 2)}{\lambda^2(\xi + C_1)} \right), \quad (9)$$

when $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$\varphi(\xi) = \ln(\xi + C_1), \quad (10)$$

when $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = \ln \left(\frac{\sqrt{4\mu - \lambda^2} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right), \quad (11)$$

and

$$\varphi(\xi) = \ln \left(\frac{\sqrt{4\mu - \lambda^2} \cot \left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right), \quad (12)$$

where a_m, \dots, λ, μ are constants to be determined later,
Step 3. After we determine the index parameter m , we substitute Eq. (4) along Eq. (5) into Eq. (3) and collecting all the terms of the same power $\exp(-m\varphi(\xi))$, $m = 0, 1, 2, 3, \dots$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i .
Step 4. substituting these values and the solutions of Eq. (5) into Eq. (3) we obtain the exact solutions of Eq. (3).

3. THE (3+1)-DIMENSIONAL YU-TODA-SASA-FUKUYAMA EQUATION

Let us consider the (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama Equation [30, 31]

$$-4u_{xt} + u_{xxxx} + 4u_x u_{xz} + 2u_z u_{xx} + 3u_{yy} = 0. \quad (13)$$

This equation is a potential-type counterpart of the (3+1)-dimensional nonlinear equation

$$(-4v_t + \phi(v)v_z)_x + 3v_{yy} = 0, \phi = \partial_x^2 + 4v + 2v_x + 2v_x \partial_x^{-1}, \quad (14)$$

introduced by Zhang et al [30] and Hu et al [31], while making the (3+1)-dimensional generalization from (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation [32]:

$$-4v_t + \phi(v)v_z = 0, \phi = \partial_x^2 + 4v + 2v_x \partial_x^{-1}, \quad (15)$$

as did for the KP equation from the KdV equation. Taking $v = u_x$ Eq.y2 transform into the potential-YTSE Eq. (13). By using the transformation (2) we get

$$u''' + 3u'^2 + (4\omega + 3)u' = 0. \quad (16)$$

Balancing U''' and u'^2 we get $m = 1$. So that we assume the solution be in the form

$$u(\xi) = a_{-1} \exp(\varphi(\xi)) + a_0 + a_1 \exp(-\varphi(\xi)). \quad (17)$$

Substituting Eq. (17) and its derivatives into Eq. (16) and collecting all term of the same power of $\exp(-i\varphi(\xi))$ where $(i = 0, \pm 1, \pm 2, \pm 3, \dots)$ and equating them to zero, we obtain system of algebraic equation by solving them by using Maple 16 we get

Case 1.

$$\omega = \mu - \frac{1}{4} \lambda^2 - \frac{3}{4}, a_0 = a_{-1} = 0, a_1 = 2.$$

The solutions of Eq. (16) according these values

When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$u(\xi) = \frac{-4\mu}{\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) + \lambda}, \quad (18)$$

and

$$u(\xi) = \frac{-4\mu}{\sqrt{\lambda^2 - 4\mu} \coth \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) + \lambda}. \quad (19)$$

When $\lambda^2 - 4\mu > 0, \mu = 0$,

$$u(\xi) = \frac{2\lambda}{\exp(\lambda(\xi + C_1)) - 1}. \quad (20)$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$u(\xi) = \frac{-4(\lambda(\xi + C_1) + 2)}{\lambda^2(\xi + C_1)} \quad (21)$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$u(\xi) = \frac{2}{\xi + C_1}. \quad (22)$$

when $\lambda^2 - 4\mu < 0$,

$$u(\xi) = \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \tan \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda}, \quad (23)$$

$$u(\xi) = \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \cot \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda}, \quad (24)$$

Case 2.

$$\mu = \frac{3}{4} + \omega, \lambda = a_0 = a_{-1} = 0, a_1 = 2.$$

The solutions of Eq. (16) according these values
When $\lambda^2 - 4\mu > 0, \mu \neq 0, \mu < 0$

$$u(\xi) = \frac{-4\mu}{\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) + \lambda}, \quad (25)$$

and

$$u(\xi) = \frac{-4\mu}{\sqrt{\lambda^2 - 4\mu} \coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) + \lambda}. \quad (26)$$

when $\lambda^2 - 4\mu < 0, \mu > 0$

$$u(\xi) = \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \tan\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) - \lambda}, \quad (27)$$

$$u(\xi) = \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \cot\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) - \lambda}, \quad (28)$$

Case 3.

$$\mu = \frac{3}{4} + \frac{1}{4} \lambda^2 + \omega, a_0 = a_1 = 0, a_{-1} = \frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega.$$

The solutions of Eq. (16) according these values
When $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) A, \quad (29)$$

and

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) B, \quad (30)$$

When $\lambda^2 - 4\mu > 0, \mu = 0,$

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) \frac{\exp(\lambda(\xi + C_1)) - 1}{\lambda}. \quad (31)$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0,$

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) \frac{\lambda^2 (\xi + C_1)}{2(\lambda(\xi + C_1) + 2)}. \quad (32)$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0,$

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) (\xi + C_1). \quad (33)$$

when $\lambda^2 - 4\mu < 0,$

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) C, \quad (34)$$

$$u(\xi) = \left(\frac{-3}{2} - \frac{\lambda^2}{2} - 2\omega\right) D. \quad (35)$$

Case 4.

$$\omega = 4\mu - \frac{3}{4}, a_0 = \lambda = 0, a_{-1} = -2\mu, a_1 = 2.$$

The solutions of Eq. (16) according these values
When $\lambda^2 - 4\mu > 0, \mu \neq 0, \mu < 0$

$$u(\xi) = \frac{A}{-2\mu} - \frac{+4\mu}{\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) + \lambda}, \quad (36)$$

and

$$u(\xi) = \frac{B}{-2\mu} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) + \lambda}, \quad (37)$$

when $\lambda^2 - 4\mu < 0, \mu > 0$

$$u(\xi) = \frac{C}{-2\mu} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \tan\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) + \lambda}, \quad (38)$$

$$u(\xi) = \frac{D}{-2\mu} - \frac{4\mu}{\sqrt{\lambda^2 - 4\mu} \cot\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1)\right) + \lambda}, \quad (39)$$

where

$$A = \left(\frac{-\sqrt{4\mu - \lambda^2} \tanh\left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1)\right) - \lambda}{2\mu}\right) \quad (40)$$

$$B = \left(\frac{-\sqrt{4\mu - \lambda^2} \coth\left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1)\right) - \lambda}{2\mu}\right) \quad (41)$$

$$C = \left(\frac{-\sqrt{4\mu - \lambda^2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1)\right) - \lambda}{2\mu}\right) \quad (42)$$

$$D = \left(\frac{-\sqrt{4\mu - \lambda^2} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1)\right) - \lambda}{2\mu}\right) \quad (43)$$

4. CONCLUSION

In this research, we introduce a new technique namely the extended $\exp(-\varphi(\xi))$ -expansion method for the first time to finding the exact and solitary wave solutions the (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama equation which play an important role in mathematical physics. From the suggested method we found that it introduce 24 solitary wave solutions according to 4 cases and these solutions give a wide range and more accurate than that obtained by other methods such as tanh method, $\left(\frac{G'}{G}\right)$ -expansion method, the $\exp(-\varphi(\xi))$ -expansion method and modified simple equation method. These solutions give more interpretation for the physical properties to the equation studied.

5. COMPETING INTERESTS

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

6. AUTHOR'S CONTRIBUTIONS

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied

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