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Relaxation parameters and composite refinement techniques

Sh.A. Meligy [a,*], I.K. Youssef [b,c]

[a] Department of Basic Engineering Sciences, Faculty of Engineering at Shoubra, Benha University, Cairo, Egypt
[b] Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
[c] Department of Mathematics, College of Science, Islamic University, Madinh, Saudi Arabia

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ABSTRACT

A composite refinement technique for two stationary iterative methods, one of them contains a relaxation parameter, is introduced. Four new techniques, Jacobi successive over relaxation (SOR) composite refinement (RJSOR), SOR Jacobi composite refinement (RSORJ), Gauss–Seidel (GS) SOR composite refinement (RGSSOR) and SOR with GS composite refinement (RSORGS) are compared with their classical forms. The efficient performance of the new forms is well established and confirmed through numerical example. The computational costs and the speed of convergence are considered. The decrease in the required number of iteration is established through the calculation of the spectral radius of the iteration matrices. It is illustrated that the convergence of Jacobi and Gauss–Seidel methods engage the divergence and extend the domain of convergence in the SOR method in the refinement technique. The calculations and graphs are performed by computer algebra system, Mathematica.

1. Introduction

The question of solving a large system of algebraic equations is a fundamental question in most modern modeling issues. Any linear system of equations can be written in matrix form as:

$$AX = b, \tag{1}$$

where, $A \epsilon R^{m\times m}$ is a coefficient matrix, $b \epsilon R^m$ is a known column of constants and X is the unknown vector. When the matrix of the coefficients is non-singular, the exact solution of the system (1) is denoted by $X = A^{-1}b$. It is well known that direct methods for solving such systems requires about $\frac{m^3}{3}$ operations which is not suitable for large sparse systems. Iterative methods appear to be the appropriate choice especially when the convergence of the method up to the required accuracy is achieved within m steps. One approach for the study of iterative techniques is through the splitting of the coefficient matrix A, $A = M - N$, with non-singular matrix M, [1–7]

$$MX^{[k+1]} = NX^{[k]} + b. \tag{2}$$

The spectral radius of the iteration matrix $\rho\left(M^{-1}N\right)$ is a measure of convergence of the iterative technique, the method with smaller spectral radius of its iteration matrix is known as asymptotically faster. Also, the splitting, $A = D - L - U$, where D is the diagonal part of the matrix A, and $-L$, $-U$ are the strictly lower and upper triangular parts of A,

* Corresponding author.
E-mail addresses: Shaban.mahmoud@feng.bu.edu.eg (Sh.A. Meligy), Kaoud22@sci.asu.edu.eg (I.K. Youssef).

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respectively [8–10] is used in the matrix reformulation of the standard stationary iterative techniques. We are interested in this work with three of the classical iterative methods.
Jacobi method [3,4,8–10]

$$X^{[k+1]} = D^{-1}(L+U)X^{[k]} + D^{-1}b = T_J X^{[k]} + C_J. \tag{3}$$

Gauss–Seidel method [3,4,8,9,11]

$$X^{[k+1]} = (D-L)^{-1}UX^{[k]} + (D-L)^{-1}b = T_{GS}X^{[k]} + C_{GS}. \tag{4}$$

The SOR method [2–4,7,12–15]

$$X^{[k+1]} = M^{-1}NX^{[k]} + M^{-1}\omega b = T_{SOR}X^{[k]} + C_{SOR}, \tag{5}$$

where

$$M = D - \omega L,\ \ N = (1-\omega)D + \omega U. \tag{6}$$

Stationary iterative techniques are characterized by their constant iteration matrices. In general, the iteration matrix is calculated only in the first step and used in the next consecutive steps, so from the second step, the computational costs are at most m^2 per iteration (much smaller for sparse matrices). The concept of refinement of an iterative technique is used to increase (double) the convergence speed of any convergent method. In this work, a composite refinement technique is introduced. We used to convergent iterative techniques one of them contains a relaxation parameter (the SOR). The effect of the refinement on the domain of the relaxation parameter as well as on the rate of convergence is studied. As we will see the domain of the relaxation parameter is extended and the spectral radius of the iteration matrix is reduced.

The paper is organized as follows: In Sections (2.1, 2.2), we consider the refinement of the SOR by Jacobi method and the refinement of Jacobi method by the SOR. In Sections (2.3, 2.4), we consider the refinement of the SOR by Gauss–Seidel method and the refinement of Gauss–Seidel method by the SOR. In Section 3, we consider a numerical example to illustrate and apply the theoretical treatment. In Section 4, a brief discussion about different iterative techniques and some of our results are introduced. Section 5 contains the conclusion of this manuscript.

2. Jacobi-SOR, SOR-Jacobi, GS-SOR and SOR-GS composite refinement methods

Refinement techniques are considered in many publications [16], and the references there in. We introduce the composite refinement approach in which two different iterative techniques are considered consecutively. The achievement in the speed of convergence of the refinement treatments dominates the increase in computational costs appears in the first step.
The basic idea in the refinement treatment is the use of a virtual step ($X^{[vir]}$) like the case of double sweep methods or the symmetric and unsymmetric techniques but without reversing the ordering of the equations [17–19].
The general iterative technique (2) can be written as

$$X^{[vir]} = M^{-1}NX^{[k]} + M^{-1}b, \tag{7}$$

and this virtual calculated data is used in a subsequent iteration as

$$X^{[k+1]} = M^{-1}NX^{[vir]} + M^{-1}b, \qquad k = 0, 1, 2, \ldots \tag{8}$$

which can be rearranged in the form

$$X^{[k+1]} = (M^{-1}N)^2X^{[k]} + (I + M^{-1}N)M^{-1}b, \qquad k = 0, 1, 2, \ldots \tag{9}$$

In the composite refinement, different iterative techniques in the consecutive sweeps are considered. We apply this concept on the three of the simple iterative methods: Jacobi, Gauss–Seidel and SOR methods.

2.1. Jacobi-SOR (RJSOR) composite refinement

The iterative formulation of the RJSOR method can be written in the form

$$\begin{aligned} X^{[k+1]} &= T_{RJSOR}X^{[k]} + C_{RJSOR}; \\ X^{[vir]} &= (D-\omega L)^{-1}[(1-\omega)D + \omega U]X^{[k]} + (D-\omega L)^{-1}\omega b. \end{aligned} \tag{10}$$

Where

$$\begin{aligned} T_{RJSOR} &= D^{-1}(L+U)(D-\omega L)^{-1}[(1-\omega)D + \omega U], \\ C_{RJSOR} &= D^{-1}[I + (L+U)(D-\omega L)^{-1}\omega]b, \end{aligned} \tag{11}$$

and this can be obtained by direct use of the above formulas (3) and (5).

Remark 2.1. From (3), (4), (10) and (11), we find

$$T_{RJSOR} = T_J T_{SOR} \ \ and \ \ C_{RJSOR} = T_J C_{SOR} + C_J. \tag{12}$$

Theorem 2.2. *Let $A \epsilon R^{m\times m}$ with $a_{ii} \neq 0$, $\rho(T_{SOR}) < 1$, and $\rho(T_J) < 1$. Then the composite Jacobi successive over relaxation method converges faster than the SOR and its domain of convergence is extended, i.e. the method can converge for some $\omega \notin (0, 2)$.*

Proof. Let μ_j be the eigenvalues of the Jacobi iteration matrix T_J, λ_j be the eigenvalues of the SOR iteration matrix T_{SOR} and σ_j be the eigenvalues of the composite Jacobi successive overrelaxation iteration matrix

$$\begin{aligned}
\det[T_{RJSOR}] &= \det[D^{-1}(L+U)(D-\omega L)^{-1}((1-\omega)D+\omega U)],\\
\prod_{j=1}^{m}\sigma_j &= \det[D^{-1}(L+U)]\ det[(D-\omega L)^{-1}((1-\omega)D+\omega U)],\\
|\prod_{j=1}^{m}\sigma_j| &= |(\prod_{s=1}^{m}\mu_s)\,(\prod_{k=1}^{m}\lambda_k)| = |\prod_{s=1}^{m}\mu_s\lambda_s| < |\prod_{s=1}^{m}\lambda_s|\\
&\leq (\rho(T_J))^m\,(\rho(T_{SOR}))^m < (\rho(T_{SOR}))^m,\\
\det[T_{RJSOR}] &= \det[D^{-1}(L+U)(D-\omega L)^{-1}((1-\omega)D+\omega U)]\\
&= \det[D^{-1}(L+U)]\ det[(D-\omega L)^{-1}]\ det[(1-\omega)D+\omega U],\\
&= \det[D^{-1}(L+U)]\ (1-\omega)^m,\\
&= (\prod_{s=1}^{m}\mu_s)\ (1-\omega)^m,
\end{aligned}$$

For the convergence of the composite Jacobi successive over relaxation method, we must have $(1-\omega)^m\ (\prod_{s=1}^{m}\mu_s) < 1$ but $(min|\mu_s|)^m < \prod_{s=1}^{m}\mu_s < (max|\mu_s|)^m < 1$,
then we can write

$$\begin{aligned}
(max|\mu_s|)^m\ (|1-\omega|)^m &< 1,\\
(\rho(T_J))^m\ (|1-\omega|)^m &< 1,\\
\rho(T_J)\ |1-\omega| &< 1,\\
-\frac{1}{\rho(T_J)} < 1-\omega &< \frac{1}{\rho(T_J)},\\
1-\frac{1}{\rho(T_J)} < \omega &< 1+\frac{1}{\rho(T_J)}.
\end{aligned}$$

Note: this will be the classical result $0 < \omega < 2$ if T_J is the identity due to the fact $\rho(I) = 1$, i.e. if $T_J = I$, then the inequality

$$1-\frac{1}{\rho(T_J)} < \omega < 1+\frac{1}{\rho(T_J)}$$

becomes

$$\begin{aligned}
1-\frac{1}{\rho(I)} < \omega &< 1+\frac{1}{\rho(I)},\\
0 < \omega &< 2.
\end{aligned}$$

Also, we have proved that $T_{RJSOR} = T_J T_{SOR}$. So if $T_J = I$, then $T_{RJSOR} = T_{SOR}$. Which means there is no refinement. □

2.2. SOR-Jacobi (RSORJ) composite refinement

The iterative formulation of the RSORJ method can be written in the form

$$\begin{aligned}
X^{[k+1]} &= T_{RSORJ}X^{[k]} + C_{RSORJ};\\
X^{[vir]} &= D^{-1}(L+U)X^{[k]} + D^{-1}b.
\end{aligned} \tag{13}$$

Where

$$\begin{aligned}
T_{RSORJ} &= (D-\omega L)^{-1}((1-\omega)D+\omega U)D^{-1}(L+U),\\
C_{RSORJ} &= (D-\omega L)^{-1}[\omega I + ((1-\omega)D+\omega U)D^{-1}]b,
\end{aligned} \tag{14}$$

and this can be obtained by direct use of the above formulas (3) and (5).

Remark 2.3. From (3), (5), (13) and (14), we find

$$T_{RSORJ} = T_{SOR}T_J \quad and \quad C_{RSORJ} = T_{SOR}C_J + C_{SOR}. \tag{15}$$

Theorem 2.4. *Let $A\epsilon R^{m\times m}$ with $a_{ii} \neq 0$, $\rho(T_{SOR}) < 1$, and $\rho(T_J) < 1$. Then the composite successive over relaxation Jacobi method converges faster than the SOR and its domain of convergence is extended, i.e. the method can converge for some $\omega \notin (0, 2)$.*

Proof. The proof is same as the proof of Theorem 2.2. □

2.3. GS-SOR (RGSSOR) composite refinement

The iterative formulation of the RGSSOR method can be written in the form

$$\begin{aligned} X^{[k+1]} &= T_{RGSSOR}X^{[k]} + C_{RGSSOR}; \\ X^{[vir]} &= (D-\omega L)^{-1}((1-\omega)D + \omega U)X^{[k]} + (D-\omega L)^{-1}\omega b. \end{aligned} \tag{16}$$

Where

$$\begin{aligned} T_{RGSSOR} &= (D-L)^{-1}U(D-\omega L)^{-1}((1-\omega)D+\omega U), \\ C_{RGSSOR} &= (D-L)^{-1}[I + U(D-\omega L)^{-1}\omega]b, \end{aligned} \tag{17}$$

and this can be obtained by direct use of the above formulas (4) and (5).

Remark 2.5. From (4), (5), (16) and (17), we find

$$T_{RGSSOR} = T_{GS}T_{SOR} \quad and \quad C_{RGSSOR} = T_{GS}C_{SOR} + C_{GS}. \tag{18}$$

Theorem 2.6. *Let $A\epsilon R^{m\times m}$ with $a_{ii} \neq 0$, $\rho(T_{SOR}) < 1$, and $\rho(T_{GS}) < 1$. Then the composite Gauss–Seidel successive over relaxation method converges faster than the SOR and its domain of convergence is extended, i.e. the method can converge for some $\omega \notin (0, 2)$.*

Proof. Let τ_j be the eigenvalues of the Gauss–Seidel iteration matrix T_{GS}, λ_j be the eigenvalues of the SOR iteration matrix T_{SOR} and σ_j be the eigenvalues of the composite Gauss–Seidel successive over relaxation iteration matrix

$$\begin{aligned} \det[T_{RGSSOR}] &= \det[(D-L)^{-1}U(D-\omega L)^{-1}((1-\omega)D+\omega U)], \\ \prod_{j=1}^{m}\sigma_j &= \det[(D-L)^{-1}U]\ det[(D-\omega L)^{-1}((1-\omega)D+\omega U)], \\ |\prod_{j=1}^{m}\sigma_j| &= |(\prod_{s=1}^{m}\tau_s)\ (\prod_{k=1}^{m}\lambda_k)| = |\prod_{s=1}^{m}\tau_s\lambda_s| < |\prod_{s=1}^{m}\lambda_s| \\ &\leq (\rho(T_{GS}))^m\ (\rho(T_{SOR}))^m < (\rho(T_{SOR}))^m, \\ \det[T_{RGSSOR}] &= \det[(D-L)^{-1}U(D-\omega L)^{-1}((1-\omega)D+\omega U)] \\ &= \det[(D-L)^{-1}U]\ det[(D-\omega L)^{-1}]\ det[(1-\omega)D+\omega U], \\ &= (1-\omega)^m\ \det[(D-L)^{-1}U], \\ &= (1-\omega)^m\ (\prod_{s=1}^{m}\tau_s), \end{aligned}$$

For the convergence of the composite Gauss–Seidel successive over relaxation method, we must have $(\prod_{s=1}^{m}\tau_s)\,(1-\omega)^m < 1$ but $(min|\tau_s|)^m < \prod_{s=1}^{m}\tau_s < (max|\tau_s|)^m < 1$,
then we can write

$$\begin{aligned} (max|\tau_s|)^m\ (|1-\omega|)^m &< 1, \\ (\rho(T_{GS}))^m\ (|1-\omega|)^m &< 1, \\ \rho(T_{GS})\ |1-\omega| &< 1, \\ -\frac{1}{\rho(T_{GS})} < 1-\omega &< \frac{1}{\rho(T_{GS})}, \\ 1-\frac{1}{\rho(T_{GS})} < \omega &< 1+\frac{1}{\rho(T_{GS})}. \end{aligned}$$

Note: this will be the classical result $0 < \omega < 2$ if T_{GS} is the identity due to the fact $\rho(I) = 1$, i.e. if $T_{GS} = I$, then the inequality

$$1 - \frac{1}{\rho(T_{GS})} < \omega < 1 + \frac{1}{\rho(T_{GS})}$$

becomes

$$1 - \frac{1}{\rho(I)} < \omega < 1 + \frac{1}{\rho(I)},$$
$$0 < \omega < 2.$$

Also, we have proved that $T_{RGSSOR} = T_{GS}T_{SOR}$. So if $T_{GS} = I$, then $T_{RGSSOR} = T_{SOR}$. Which means there is no refinement. □

2.4. SOR-GS (RSORGS) composite refinement

The iterative formulation of the RSORGS method can be written in the form

$$X^{[k+1]} = T_{RSORGS}X^{[k]} + C_{RSORGS}; \quad X^{[vir]} = (D-L)^{-1}UX^{[k]} + (D-L)^{-1}b. \tag{19}$$

Where

$$T_{RSORGS} = (D-\omega L)^{-1}((1-\omega)D + \omega U)(D-L)^{-1}U, \quad C_{RSORGS} = (D-\omega L)^{-1}[\omega I + ((1-\omega)D + \omega U)(D-L)^{-1}]b, \tag{20}$$

and this can be obtained by direct use of the above formulas (4) and (5).

Remark 2.7. From (4), (5), (19) and (20), we find

$$T_{RSORGS} = T_{SOR}T_{GS} \;\; \text{and} \;\; C_{RSORGS} = T_{SOR}C_{GS} + C_{SOR}. \tag{21}$$

Theorem 2.8. *Let $A \epsilon R^{m\times m}$ with $a_{ii} \neq 0$, $\rho(T_{SOR}) < 1$, and $\rho(T_{GS}) < 1$. Then the composite successive over relaxation Gauss–Seidel method converges faster than the SOR and its domain of convergence is extended, i.e. the method can converge for some $\omega \notin (0, 2)$.*

Proof. The proof is same as the proof of Theorem 2.6. □

3. Numerical example

In [20] a good numerical example is considered to compare the performance of Jacobi, Gauss–Seidel and SOR methods. We use this example to extend the comparisons with RSORJ, RJSOR, RSORGS and RGSSOR methods.

Example 3.1. We consider the linear system of equations, [20]

$$\begin{aligned} -4.2x_1 + x_3 + x_4 + x_7 + x_8 &= -6.2,\\ x_1 - 4.2x_2 + x_4 + x_5 + x_8 &= -5.4,\\ x_1 + x_2 - 4.2x_3 + x_5 + x_6 &= 9.2,\\ x_2 + x_3 - 4.2x_4 + x_6 + x_7 &= 0,\\ x_3 + x_4 - 4.2x_5 + x_7 + x_8 &= -6.2,\\ x_1 + x_4 + x_5 - 4.2x_6 + x_8 &= -1.2,\\ x_1 + x_2 + x_5 + x_6 - 4.2x_7 &= 13.4,\\ x_2 + x_3 + x_6 + x_7 - 4.2x_8 &= -4.2, \end{aligned} \tag{22}$$

with exact solution is $X = (1, 2, -1, 0, 1, 1, -2, 1)^T$.

4. Results and discussions

The stationary iterative techniques for solving a linear system (1) are

$$X^{[k+1]} = M^{-1}NX^{[k]} + M^{-1}b.$$

Jacobi, Gauss–Seidel and the SOR are the standard examples for such methods. There are methods such as MSOR, AOR, MPAOR, SSOR, USOR which include more parameters than the SOR or use the concept of double sweep with different attitudes. In any case there is no doubt that for large sparse systems iterative techniques are more efficient than any

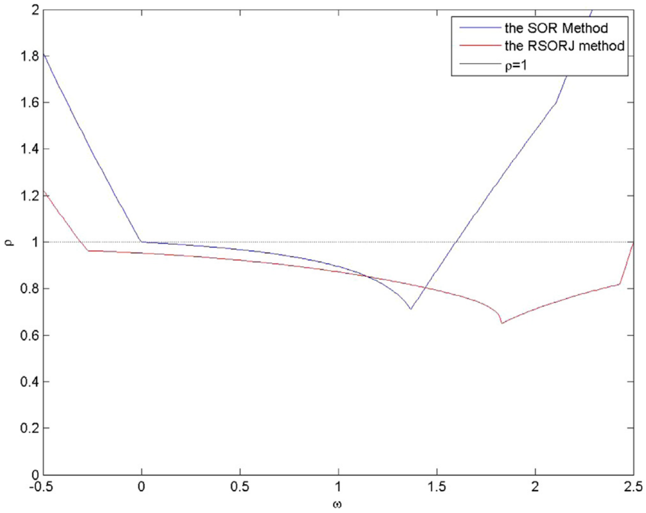

Fig. 1. The behavior of the spectral radius for the SOR and RSORJ methods.

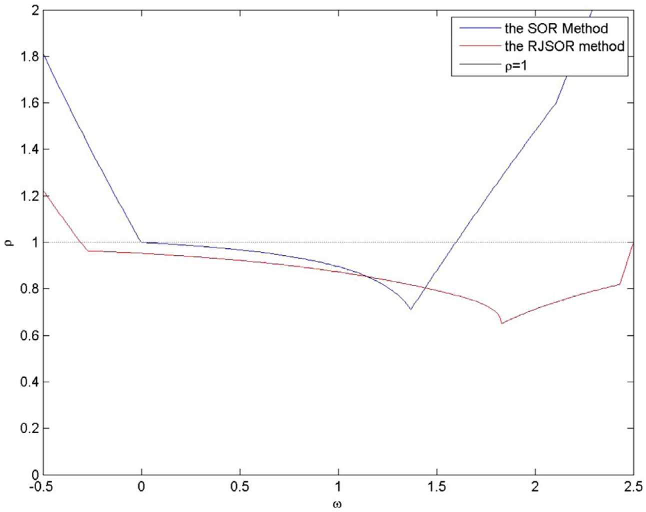

Fig. 2. The behavior of the spectral radius for the SOR and RJSOR methods.

Table 1
Comparison of the spectral radius at different values of the relaxation parameter.

$\rho(T_J) = 0.952381$			$\rho(T_{GS}) = 0.895304$		
ω	$\rho(T_{SOR})$	$\rho(T_{RSORJ})$	$\rho(T_{RJSOR})$	$\rho(T_{RSORGS})$	$\rho(T_{RGSSOR})$
−0.3	1.47073	0.991042	0.991042	0.898256	0.898256
1.4	0.750677	0.811681	0.811681	0.679757	0.679757
1.7	1.12563	0.745848	0.745848	0.510316	0.510316
1.83	1.28151	0.649842	0.649842	0.401246	0.401246
1.85	1.30513	0.658163	0.658163	0.413514	0.413514
2.3	2.01941	0.791626	0.791626	0.821105	0.821105
2.4	3.15701	0.812437	0.812437	0.937399	0.937399

other direct technique. In iterative techniques advantages of existence of initial estimations of the expected solution can be considered to reduce the computational costs, which is the case in most realistic applications [11,21,22]. Iterative techniques require a small storage in comparison with direct methods. Moreover, the storage requirements can be easily predicted in advance. The most vital disadvantage of iterative techniques is their slow rate of convergence. The spectral radius of the iteration matrix of the iterative technique is taken as a measure for the rate of convergence of the method. Jacobi and Gauss–Seidel methods have constant spectral radii for each problem. The spectral radius of the iteration matrix

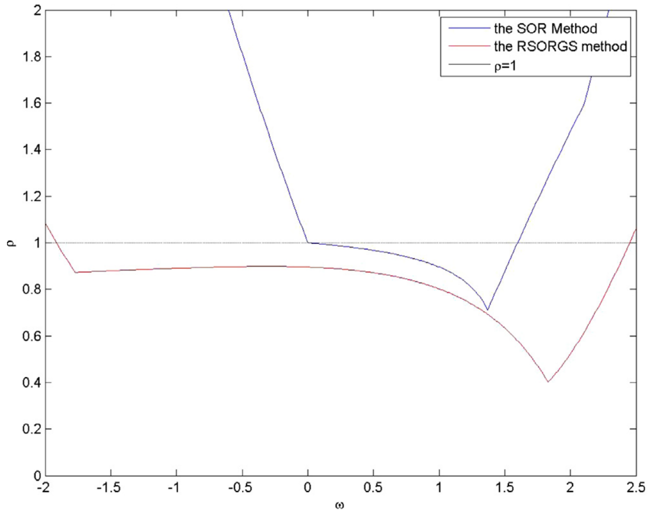

Fig. 3. The behavior of the spectral radius for the SOR and RSORGS methods.

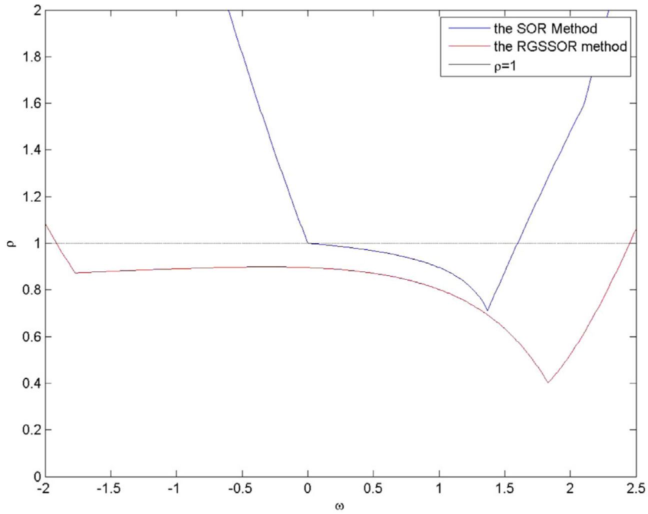

Fig. 4. The behavior of the spectral radius for the SOR and RGSSOR methods.

Table 2
The solution of the linear system (22) by using the RSORJ method at $\omega = 1.85$.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$	$x_5^{[k]}$	$x_6^{[k]}$	$x_7^{[k]}$	$x_8^{[k]}$
0	0	0	0	0	0	0	0	0
1	−0.453515	2.17665	−0.655397	−0.609401	0.0457753	0.237839	−2.34687	0.741098
2	0.405863	2.20524	−0.991473	−0.463579	0.590586	0.973657	−1.73893	1.31651
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	1	2	−0.999999	9.22×10^{-7}	1	1	−2	1
38	1	2	−1	5.20×10^{-7}	1	1	−2	1

of the SOR method $\rho(T_{SOR})$ depends on the choice of $0 < \omega < 2$. There is an optimum value ω_{opt}, for each convergent problem, corresponding to the minimum value of the spectral radius of the considered iteration matrix. One of the difficulties of using the SOR technique is the choice of ω_{opt}. The major step in such iterative techniques (stationary iterative techniques) is the first step in which the iteration matrix of the method is calculated and used in any subsequent step. A refinement technique is a technique in which each step is equivalent to two steps of the original method using a virtual (hidden) step (7). In the composite refinement technique, two different iterative techniques are used (10, 13, 16 and 19), so the convergence rate is faster than any of its components (Theorems 2.2 and 2.6). The interesting results in this work that, when a composite refinement of the SOR method with one of the simple methods (Jacobi or Gauss–Seidel) not only the rate

Table 3
The solution of the linear system (22) by using the RJSOR method at $\omega = 1.85$.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$	$x_5^{[k]}$	$x_6^{[k]}$	$x_7^{[k]}$	$x_8^{[k]}$
0	0	0	0	0	0	0	0	0
1	2.27628	3.78148	0.729615	1.2244	2.27628	2.78148	−0.270385	2.2244
2	1.65437	2.9855	−0.007085	0.67596	1.65437	1.9855	−0.992915	1.67596
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	1	2	−0.999999	-3.81×10^{-7}	1	0.999999	−2	1
38	1	2	−1	-2.42×10^{-7}	1	1	−2	1

Table 4
The solution of the linear system (22) by using the RSORGS method at $\omega = 1.83$.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$	$x_5^{[k]}$	$x_6^{[k]}$	$x_7^{[k]}$	$x_8^{[k]}$
0	0	0	0	0	0	0	0	0
1	0.358437	2.01463	−0.873532	0.007485	0.870564	0.645723	−2.51848	0.849314
2	0.779889	1.68616	−1.29227	−0.194371	0.872178	0.86007	−2.21495	0.703561
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
16	0.999999	2	−1	2.94×10^{-9}	1	1	−2	0.999999
17	1	2	−1	-3.75×10^{-7}	1	1	−2	1

Table 5
The solution of the linear system (22) by using the RGSSOR method at $\omega = 1.83$.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$	$x_5^{[k]}$	$x_6^{[k]}$	$x_7^{[k]}$	$x_8^{[k]}$
0	0	0	0	0	0	0	0	0
1	2.20082	3.59406	0.571015	1.62171	2.79886	2.80447	−0.476615	2.54594
2	2.35878	3.2798	0.204841	1.16216	2.14555	2.15959	−0.822922	2.14793
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	1	2	−0.999999	5.01×10^{-7}	1	1	−2	1
19	1	2	−1	1.75×10^{-7}	1	1	−2	1

of convergence is increased but also the domain of convergence is extended (Theorem 2.2, Theorem 2.6 and Table 1). The optimum value ω_{opt} is slightly shifted and the corresponding spectral radius is reduced (Table 1 and (Figs. 1–4)). Table 1 and (Figs. 1–4) illustrate the performance of the four formulas RSORJ, RJSOR, RSORGS, RGSSOR in comparison with the SOR. Seven different values of ω are selected, the SOR is divergent at six values while the four methods are convergent at all the selected values. Also, the spectral radius of the SOR method is greater than the spectral radius of any other of the refinement forms (both of Jacobi and Gauss–Seidel methods are convergent). The four formulas RSORJ, RJSOR, RSORGS, RGSSOR are convergent at some values outside side the standard domain in the SOR method $0 < \omega < 2$. It is interesting to notice that although both of RSORJ and RJSOR have the same behavior, the calculated values at intermediate steps are different, (Tables 2 and 3). The same behavior is noticed for RSORGS and RGSSOR, (Tables 4 and 5).

5. Conclusion

The composite refinement techniques for two stationary iterative methods, one of them contains relaxation parameter, are presented. The composite refinement techniques are the appropriate choice for solving a large system of algebraic equations. (Theorems 2.2, 2.4, 2.6 and 2.8) prove that the new four methods (RJSOR, RSORJ, RGSSOR and RSORGS) increase the rate of convergence, decrease the number of iterations in comparison with the classical methods. The (Figs. 1–4) illustrate that the spectral radius behavior of the methods (RJSOR, RSORJ, RGSSOR and RSORGS) with respect to the SOR method. The efficient performance of the new techniques is illustrated through the numerical Example 3.1 as shown in (Tables 2 and 3). Although the solution of the linear system (22) is divergent by SOR method at the values illustrated in Table 1, the solution of this system is convergent by the RSORJ, RJSOR, RSORGS and RGSSOR methods. Therefore, the new techniques achieve a qualitative shift in solving large linear system of algebraic equations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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