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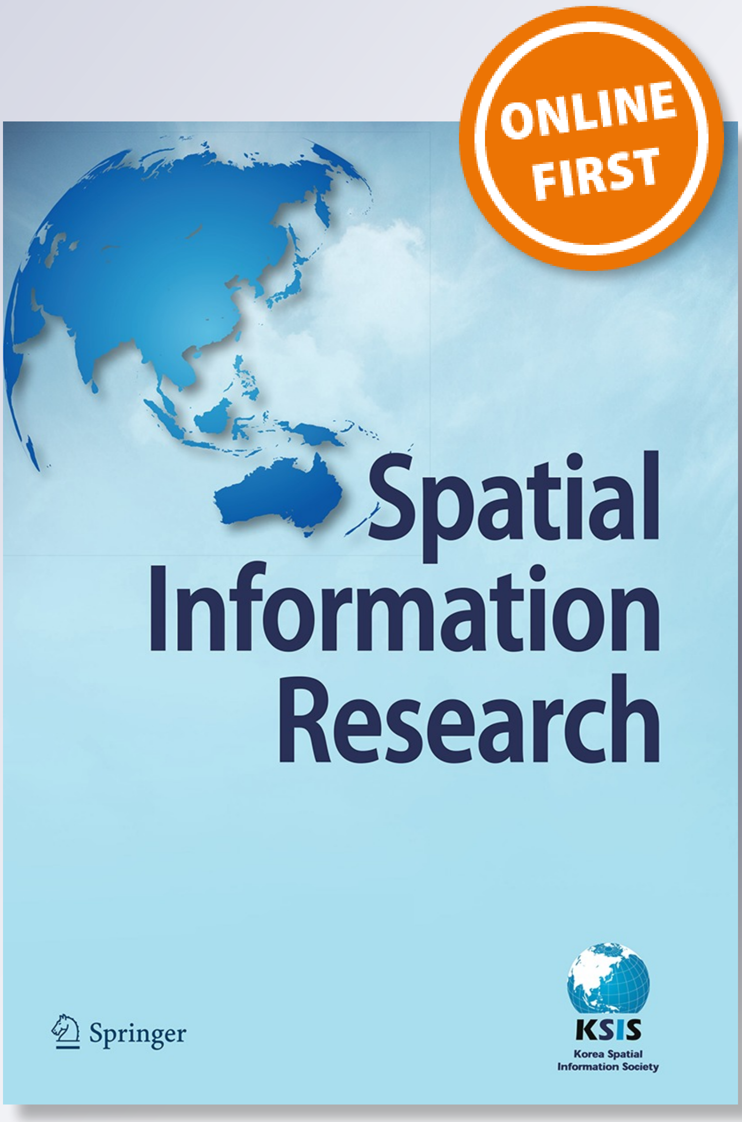

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Comparative evaluation among various robust estimation methods in deformation analysis

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Abstract There is different robust estimation methods used for detecting small gross errors which may be presence in geodetic observations. Some of these methods are robust M-estimators, Least Absolute Sum and Danish method. The conventional Least Squares Estimation method and these robust estimation methods have been tested and applied on a precise geodetic network designed for detecting earth's crustal deformations using single-point and rigid body displacement models. The results show that, the method of least squares leads to biased, unfavorable solution and useless estimates if the single-point displacement does not taken into considerations in the solution's mathematical model. In contrast, the results of robust estimation methods are attained unbiased in a single estimation step. A reliable separation of single-point displacement and the general deformation model is achieved without difficulties. The comparison of results proved that, the Danish method gave the most accurate results. It also was more sensitive to outliers which may be presence in surveying observations in case of applying single-point displacement model. The Least Absolute Sum method was very nicely reproduces the simulated deformations and does not show a contamination of the estimated positions of stable points in the case of rigid body displacement model. The results of adjustment using rigid body displacement model were not identical with the nearly correct results of single-point displacement model. So, the rigid body displacement model may be not suitable for geodetic networks.

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Keywords Gross errors · Robust M-estimators · Least absolute sum (LAS) · Danish method · Deformation analysis

1 Introduction

Deformation analysis with geodetic methods is based on the comparison of two or more sets of coordinates of a group of points representing an object required to be investigated. Each set of coordinates is estimated independently in a mathematical model from geodetic observations pertaining to a specific epoch of time. Thus, the epoch differences of coordinates reflect all deformations of the object. The parameters models are usually estimated by the least squares method, and subsequently, under the assumption of normality, statistical tests are used to investigate the significance of parameters and the adequacy of the model [1]. The approach is optimal, in the statistical sense, if the observations are normally distributed. However, in practice, assumptions are incorrect. In this research, different robust estimation methods are tested in the adjustment model. These methods have clear advantages over least squares in the presence of unordered effects such as gross errors or single-point displacements [2]. Robust estimation procedures are insensitive to small deviations from the assumptions model. They have the desired property that local deviations do not contaminate the whole residual vector but inflate the corresponding residuals [2].

Various robust estimation methods and deformation models has been applied on precise geodetic network. Such methods used by Caspary et al. [4] which apply the robust M-Estimation method to deformation analysis which proved that this method is insensitive to very small outliers. Another study is done by Dielman et al. [7] in which the LAV (Least Absolute Value) Estimation method in Linear Regression is applied.

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2 Deformation model

Under the assumptions that the selected datum and the approximate coordinates of the identical points are the same in both epochs, the general deformation model can be written as follows [2]:

$$\Delta = \bar{x}_2 - \bar{x}_1 = Ht + \delta \, , \; E(\Delta) = Ht \tag{1}$$

where $\bar{x}_2, \bar{x}_1$ are the vectors of estimated coordinates of the two epochs, H is the design matrix, t is the vector of deformation parameters, δ is the vector of residuals and $E(.)$ is the statistical expectation.

2.1 Model of single-point displacements

A similarity transformation is carried out as a first step in the case of no a priori knowledge about the possible deformations or if the analysis is carried out to investigate the stability of points in a reference net. The objective is to detect single-point movements by screening the residuals of the transformation. The parameter vector t of the similarity transformation contains 4 parameters. These parameters are translation along x-axis (t_x), translation along y-axis (t_y), rotation about z-axis and scale parameter (s). The design matrix $H^t = (H_1^t, H_2^t, \ldots, H_p^t)$consists of p sub-matrices referring to the p common points of epochs. All sub-matrices have the same pattern as [2]:

$$H_i = \begin{vmatrix} 1 & 0 & -y_i & x_i \\ 0 & 1 & x_i & y_i \end{vmatrix}, \quad i \in \{1, 2, \ldots, p\} \tag{2}$$

where x_i, y_i are the coordinates of point P_i w.r.t. the center of gravity of the set of points. The vectors $\Delta = (\Delta_1^t, \Delta_2^t, \Delta_3^t, \ldots, \Delta_p^t)^t$ and $\delta = (\delta_1^t, \delta_2^t, \delta_3^t, \ldots, \delta_p^t)^t$are partitioned conformable to Eq. (2), so that the components refer to the individual points, for example $\Delta_i = (\Delta x_i, \Delta y_i)^t$ and $\delta_i = (\delta x_i, \delta y_i)^t$ refer to P_i.

2.2 Model of rigid body displacements

A simple modification of Eq. (1) suffices to model a situation of network separated in different parts by, for instance, faults where a relative shift of one part with respect to another is anticipated. The design matrix H is partitioned conformably to blocks and an individual set of deformation parameters is allocated to each block. For instance, the following functional model can be applied for two blocks and easily be extended to three or more blocks [3]:

$$\Delta = \begin{vmatrix} H_1 & 0 \\ 0 & H_2 \end{vmatrix} \begin{vmatrix} t_1 \\ t_2 \end{vmatrix} + \delta \tag{3}$$

The relative rigid body displacement is parameterized by the components of $\Delta t = t_2 - t_1$. The vector of residuals δ contains all effects being not modeled, especially single-point movements which do not conform to the trend.

3 Parameter estimation

The conventional least squares method leads to unbiased estimates with minimum variance if the model is entirely correct [4]. But in the case of observations contain gross errors or single-point displacement, the outcome is no longer optimal. These blunders or displacements contaminate the estimates of parameters and are spread over the whole residual vector. So that it is extremely difficult and sometimes impossible to locate blunders by screening the residuals. This disadvantage can be avoided, if a robust estimation method is applied. Robust estimates are not influenced by blunders as long as the majority of points conform to the modeled trend [1]. Three robust estimation methods suitable for the parameter estimation in the presence of deviations and consistently superior to least squares will be presented in the next subsections.

3.1 Robust M-estimators

The conventional maximum likelihood estimators are derived for well-defined distribution functions f (x, θ), leading for instance to the normal distribution of the least squares method as follows [5]:

$$\sum p_i v_i^2 = \sum p(x_i, t) \rightarrow \min \tag{4}$$

where p_i is the weight of observation x_i and v_i is the corresponding residual.

The parameters are denoted by θ and their estimate by t. Equation (1) is solved for t by equating the sum of the derivatives to zero as follows [6]:

$$\Psi(x, \theta) = \frac{\partial(p(x, \theta))}{\partial\theta}, \quad \sum \Psi(x_i, t) = 0 \tag{5}$$

Robust M-estimators are in a certain way generalization of maximum likelihood estimators. The derivation starts by defining a suitable function $p(x_i, t)$ to be minimized in the same way as outlined above. The application of the general deformation model (Eq. 1) leads to the following target function [7]:

$$\sum p(\Delta - H_i t) = \sum p(\delta_i) \rightarrow \min \tag{6}$$

and after differentiation:

$$\sum H_i^t \Psi(\delta_i) = 0 \tag{7}$$

The computation of parameters requires the solution, in general, of non-linear set of Eq. (7) which can be represented as follows:

$$w_i = \frac{\Psi(\partial_i)}{\partial_i}, \quad W = diag(w_1, w_2, \ldots, w_{2P}) \tag{8}$$

Equation (7) can be written in matrix notation as follows:

$$\sum H^t W \delta = 0 \tag{9}$$

The analogy of Eq. (9) with the equations of least squares estimation immediately leads to the following iteration scheme [7]:

$$\left.\begin{array}{l} t_v = (H^t W_v H)^{-1} H^t W_v \Delta \\ (w_i)_{v+1} = \dfrac{\Psi(\Delta_i - H_i t_v)}{\Delta_i - H_i t_v} \\ (W_i)_{v+1} = diag((w_1), (w_2), \ldots, (w_{2p}))_{v+1} \\ W_1 = I, \quad v = 1, 2, \ldots \end{array}\right\} \tag{10}$$

This equation is known as iteratively reweighted least squares method [1]. According to the proposal of Huber [6], the following function has been selected for the numerical examples:

$$p(\delta_i) = \begin{cases} \delta_i^2 & \text{for } |\delta_i| \leq c \\ c|\delta_i| - \dfrac{c^2}{2} & \text{for } |\delta_i| \rangle c \end{cases} \tag{11}$$

with derivatives and weights, we get

$$\Psi(\delta_i) = \begin{cases} \delta_i \\ c \end{cases}, \quad w_i = \begin{cases} 1 & \text{for } |\delta_i| \leq c \\ \dfrac{c}{|\delta_i|} & \text{for } |\delta_i| > c \end{cases} \tag{12}$$

For a distribution which is Gaussian between +c and double exponential outside this region, Eq. (10) establishes a maximum likelihood estimator. For the computations, the model of Eq. (12) has been used with the constant c = 2σ. The objective functions are minimized in the computational process. This function has, for equal a priori weights $w_1 = 1$, the slightly simplified form is given by [4]:

$$\sum p(\delta_i) \rightarrow \min,$$

$$p(\delta_i) = \begin{cases} \dfrac{\delta_i^2}{2} & \text{for } \delta_i \leq c \\ c\delta_i - \dfrac{c^2}{2} & \text{else} \end{cases} \tag{13}$$

The weight function is given by:

$$w_i = \begin{cases} 1 & \text{for } \delta_i \leq c \\ c/|\delta_i| & \text{else} \end{cases} \tag{14}$$

3.2 The least absolute sum (LAS) method

The estimation principle of minimizing the sum of the absolute values of residuals has been known since long time ago, but it only became feasible with the advent of modern computers. Brunner [8] showed that this principle is not sensitive to observations contain outliers, and one can easily see that it is a special case of an M-estimator. To adapt this principle to the problem of observations containing outliers, it has been modified by replacing the absolute values of the individual residuals by the lengths of the discrepancy vectors of the points [7] as follows:

$$\sum d_j \rightarrow \min \tag{15}$$

With,

$$d_j^2 = \delta_j^t \delta_j, \quad \delta_j^t = [\delta_{x_j}, \delta_{y_j}] \quad \text{and} \quad j = 1, 2, 3, \ldots p$$

Then,

$$\frac{\partial}{\partial t} \sum_{j=1}^{p} [(\Delta_j - H_j t)^t (\Delta_j - H_j t)]^{0.5} = 0 \tag{16}$$

After differentiation, we get:

$$\sum_{j=1}^{p} \frac{H_j \delta_j}{d_j} = 0 \tag{17}$$

Or equivalently in

$$H^t D^{-1} H^t - H^t D^{-1} \Delta = 0 \tag{18}$$

with, $D = diag[d_1, d_1, d_2, d_2, \ldots, d_p, d_p]$.

The numerical computation of the parameter vector t and of the discrepancies d_j is again carried out by a kind of iteratively reweighted least squares process using the iteration scheme as follows:

$$\left.\begin{array}{l} t_{v+1} = (H^t D_v^{-1} H)^{-1} H^t D_v^{-1} \\ \delta_{v+1} = H t_{v+1}, \quad v = 1, 2, 3, \ldots \end{array}\right\} \tag{19}$$

If the computation is started with $D_1 = I$, one gets in the first step the least squares estimate of t. To avoid numerical problems, if some residuals become very small, d_j is replaced by $d_j + \varepsilon$ for a suitably small ε. The objective function is given by [4]:

$$\sum \delta_j \rightarrow \min \quad \text{and} \quad d_j^2 = \delta x_j^2 + \delta y_j^2$$

The weight function is given by:

$$w_j = d_j^{-1} \quad \text{and} \quad d_j^2 = \delta x_j^2 + \delta y_j^2 \tag{20}$$

3.3 The Danish method

The Danish method has been developed to detect gross errors in the vector of observations in geodetic and photogrammetric adjustment models [2, 9]. It has no clear statistical meaning, but it has proved that it is an extremely useful heuristic tool of data screening. After a conventional least squares adjustment, new weights are computed from the residuals. The applied equations are tailored to suit the problem at hand. The following are the weight functions of the Danish method:

$$w_{v+1} = w_v f(\delta_v), \quad v = 1, 2, 3, \ldots$$

$$f(\delta) = \begin{cases} 1 & \text{for } \dfrac{|\delta|\sqrt{w_1}}{c\sigma} \langle c \\ \exp - \left(\dfrac{|\delta|\sqrt{w_1}}{c\sigma}\right) & \text{else} \end{cases} \quad (21)$$

with these weights, the least squares computation is repeated, leading to new residuals by virtue of 20 new weights. This process of reweighting and adjustment is repeated until convergence is achieved. The weights of outlying observations become small, thus being of minor influence on the parameter estimation. In this sense, the method can be classified as a robust one in respect of gross residuals. For the following numerical examples in case of observations contains outliers, a priori weights $w_1 = 1$ for all points and the constant $c = 2\ \sigma$ have been selected. Since outlying coordinates may indicate that the observations contain gross errors, these should be detected easily by the Danish method. The objective function of this method is given by [4]:

$$p(\delta_i) = \begin{cases} \dfrac{\delta_i^2}{2} & \text{for } |\delta_i| \leq c \\ c^2 + c|\delta_i| \exp - \dfrac{|\delta_i|}{c} & \text{else} \end{cases} \quad (22)$$

The least squares principle minimizes given by: $\sum \delta_i^2 \rightarrow \min$

The least squares, M-estimation and Danish methods behave the interval $\pm c$. Observations with residuals outside this interval are weighted down by Eq. (22) more drastically than by Eq. (15). The numerical computations for all methods can be carried out by an iteratively reweighted least squares mathematical model (Eqs. 10, 21). The weight functions are given by:

$$w_i = \begin{cases} 1 & \text{for } \delta_i \leq c \\ \exp\left(-\dfrac{|\delta_i|}{c}\right) & \text{else} \end{cases} \quad (23)$$

The weight function of least squares method is given by:

$w_i = 1$ independent of δ_i.

4 Test of significance

In robust estimation methods, the relationship between observations and estimated parameters is non linear and can not linearized in the usual way [3], then the variance propagation law could not be applied directly. The variance–covariance matrices of the parameters and residuals are required for significance testing. Under the assumption of normality, the parameter vector can be tested for significance using the χ^2 distribution. The null hypothesis [1] $H_o{:}E(t) = 0$ is rejected with the type I error probability of α percent if the statistic $T = t^t \sum_t^{-1} t$ is $\leq$ to the critical value $\chi^2_{m,1-\alpha}$ which is equivalent to the following probability relation: $P\{T \leq \chi^2_{m,1-\alpha}\} = 1 - \alpha$ in which m is the number of elements t. For detecting deformation, the residuals δ_j are tested for significance. The components δx and δy of points are treated simultaneously. The null hypothesis $H_o{:}E(\delta_j) = 0$, $\delta_j = (\delta x_j, \delta y_j)^t$ is tested for all points. Since these tests are stochastically dependent, it is not possible to develop a statistically rigorous procedure. The probability relation based on the χ^2 distribution of the test statistic $T_j = \delta_j^t \sum_j^{-1} \delta_j$ holds true only for single test. The type I error of rejecting H_o in the simultaneous test of p points remains unknown. Nevertheless, the critical value $\chi^2_{2,1-\alpha}$ for $\alpha = 5\ \%$, for instance, leads to plausible decision. This is one of disadvantages of the robust estimation. Deviations are not spread over the whole residual vector, then they can be detected and contaminate the estimates of parameters. It is not required to repeat the adjustment after discarding outliers in observations because the observations and their residuals would not change.

5 Numerical example and results

A monitoring network (see Fig. 1) is designed to compare the results of different estimate principles in deformation models. The network consists of 12 points has been used to

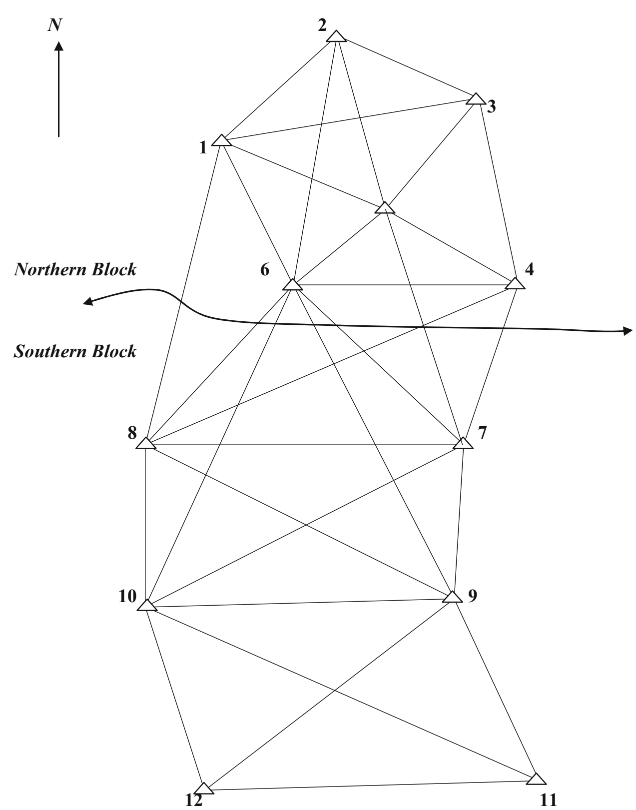

Fig. 1 The designed geodetic network

illustrate the properties of robust estimation in both single-point displacements and rigid-body displacement models. Distances and directions between points have been observed. The observations are normally distributed directions with $\sigma_R = 0.4''$ and distances with $\sigma_D = 0.5$ mm. The free adjustment technique is applied on the designed network and observations. For this purpose, a computer program using C ++ programming language is designed and applied on the real observation. The horizontal displacements in x- and y-direction (dx and dy) has been computed by the difference of x- and y-coordinates obtained from two successive epochs.

Gross errors simulation: gross errors of constant size were injected into the distances those have been observed from point 3. The sizes of the simulated gross errors were inserted as 5 mm. The simulated outliers were randomly added to the distances observed from this point. Tables 1 and 2 show the estimated values of displacements. The statistics T_j are computed to be indicators for the significance of these displacements. The critical value for a simultaneous test with $\alpha = 5$ % is $\chi^2_{2,0.95} = 6.0$, while the corresponding value for a simultaneous test for all network points neglecting their dependency, with an overall type I error probability of 5 % is 11.2. The unknown true critical value for $\alpha = 5$ % of these simultaneous tests is somewhere between these limits.

Table 1 The LS estimated displacements of all network points (dimension in mm)

Point no.	Estimated displacements					
	LS			Robust M-estimators		
	dx	dx	T_j	dx	dy	T_j
1	8.07	8.01	■41.18	4.43	4.00	■12.33
2	7.36	7.56	■6.20	4.00	4.03	1.98
3	9.33	8.98	■97.26	5.78	4.98	■84.7
4	8.45	7.78	■11.40	5.00	4.76	0.18
5	8.44	7.74	■33.21	4.76	4.00	2.90
6	7.45	6.65	2.17	4.00	4.00	1.52
7	7.90	7.95	8.50	4.01	3.99	4.44
8	7.72	7.91	4.18	4.12	3.00	3.32
9	5.77	5.02	9.10	2.00	2.98	4.74
10	5.65	5.23	2.15	2.87	2.65	4.23
11	5.27	6.87	1.23	2.98	3.00	3.19
12	6.45	6.76	3.57	2.01	3.05	2.11

Values of Tj with symbol ■ exceed the critical value of the significance test

LS Least Squares, dx and dy are horizontal displacements

5.1 Analysis of results

5.1.1 Single-point displacements

- The observations were with normally distributed numbers with $\sigma_R = 0.3''$ for directions and $\sigma_D = 0.3$ mm for distances.
- The average of the mean point error is about 0.3 mm.
- The generation of the second epoch includes small displacements in points 8, 9, 10, 11 and 12.
- The coordinate differences were analyzed applying the deformation model of Eq. (1) and the estimation methods (LS, M-estimators, LAS and Danish).

Results of applying LS

The results of applying least squares method (LS) show that (see Table 1):

- The displacement of point 3 at which all distances contain gross errors corrupts all other estimates.
- The results of adjustment indicate that, points 8, 9, 10, 11, and 12 have been displaced or moved by a relatively large amount range from 5 mm to 7 mm. This is due to that these points lie relatively far from point 3.
- On the other hand, points 1, 2, 3, 4, 5, 6 and 7 have been displaced or moved by larger amount reach to 9 mm. This is due to that these points connected directly with point 3.
- The statistics T_j of points from 1 to 5 exceed the lower critical value. This is an indicator to that these points have been affected directly by the gross errors that was injected in observations taken at point 3.

Table 2 The estimated displacements of all network points obtained from different Robust weighting functions (dimension in mm)

Point no.	LAS			*Danish*		
	dx	dy	T_j	dx	dy	T_j
1	3.98	3.03	0.39	3.12	2.65	0.00
2	3.56	3.14	1.51	3.00	2.57	1.31
3	4.87	4.00	■64.22	4.34	3.00	■49.67
4	4.03	4.00	0.00	3.56	3.23	0.00
5	4.00	3.02	1.19	3.50	2.34	1.17
6	3.12	3.05	1.32	2.45	2.34	0.92
7	3.34	3.00	3.01	2.67	2.15	0.46
8	3.34	3.00	3.00	1.14	2.34	2.55
9	2.00	2.00	5.25	1.34	1.12	2.50
10	2.01	1.87	3.65	1.45	1.00	2.11
11	2.13	2.01	2.76	1.32	1.00	4.64
12	1.98	2.45	5.23	1.23	1.76	4.34

LAS least absolute sum

- Further localization tests and recursive estimation procedures are required to separate the true from the apparent displacements. Finally, the displaced points are excluded; the last results depend on the outcome of the localization tests.

Results of applying robust methods

The results of applying robust estimation methods show that (see Table 2):

- The residuals of robust estimation methods are less influenced by the displacement of point 3.
- The statistics T_j of point 1 and 3 are the only points exceed the lower critical value. The T_j of All robust methods for the other points do not exceed the lower critical value.
- The robust estimates of deformations parameters model do not require repeated localization or estimation procedures because the final displacements are determined in only one step independent of any test results.
- The results of adjustment indicate that some points may be not displaced.
- A reliable separation of single-point displacements and the general deformation model is achieved without difficulties.
- The statistics T_j of the Danish method at all points are less than their corresponding values obtained from the other robust methods.
- 80 % from the gross errors value could separate from observations in M-estimators method while around 87 % in LAS and more than 95 % could separate in Danish method.
- Since the results for the robust methods are very similar, the further computations could be restricted to one method. The Danish method is chosen to be this method due to its geometric evidence in deformation analysis.

Figure 2 shows the weights of the Least Squares (LS), Robust M-estimation, LAS and Danish methods against the residuals δ_i. The figure shows that the most robust and sensitive method to outlying observations is the Danish method followed by LAS estimation.

5.1.2 Rigid body displacements

- The first simulated relative rigid body displacements of the northern block of the network by $\Delta x = 4$ mm and $\Delta y = 5$ mm and larger at points 4 and 6.
- The first simulated relative rigid body displacements of the southern block of the network by $\Delta x = 17$ mm and $\Delta y = 19$ mm.
- The difference between LS and LAS results are clear as shown in Tables 3 and 4.

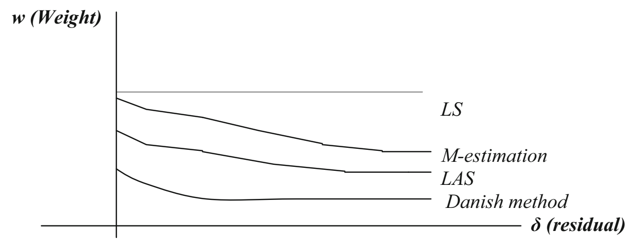

Fig. 2 Weight functions of different estimation methods

- The critical value of the test statistics T_j for a type I error of $\alpha = 5$ % lies between 6.0 and 11.6 as defined in this research.
- The LAS method very nicely reproduces the simulated deformations and does not show a contamination of the estimated positions of stable points.
- In contrast, the LS results are simply useless.
- Nearly, all points show apparent displacements and the rigid body shift parameters are estimated poorly.
- Several iterative steps of testing and estimating are necessary to achieve a reasonable result.
- 95 % from the gross errors value could separate from observations in LAS method.
- The results of adjustment using rigid body displacement model are not identical with the nearly correct results of single-point displacement model.
- So, the rigid body displacement model may be not suitable for geodetic networks.

6 Conclusion

The test of this research confirms the favorable property of robust estimation methods in case of mathematical model contains deviations. Single-point displacements can not be detected prior to the analysis and can not account in deformation model. Application of Least Squares (LS) method for estimating deformation parameters led to useless results if the observations contain blunders. Application of robust methods led to more correctly outcome and the small single-point displacements are detected easily. In deformation analysis, unbiasedness of parameters and detectability of single-point displacements are of primary interest. Then, robust estimation methods should be taken into account as an alternative or as a supplement to the classical LS method. The differences between LS and LAS results were clear in case of using rigid body displacement model. The LAS method very precisely reproduces the simulated deformations and did not show a contamination of the estimated positions of stable points. Rigid body shift parameters are estimated poorly. In deformation analysis, unbiasedness of parameters and detectability of single

Table 3 The simulated rigid body and single point displacements in northern block of the network (dimension in mm)

Point no.	Simulated block shift		Estimated displacements					
			LS method			Robust M-estimators		
	dx	dy	dx	Dy	T_j	dx	dy	T_j
(a)								
1	–	–	3	4	5.10	2.9	3.9	0.03
2	–	–	4	4	5.55	3.8	3.9	0.03
3	–	–	5	4	4.85	4.7	3.8	2.01
4	–	–	10	12	■■18.68	5.7	5.8	2.66
5	–	–	4	4	4.86	3.9	1.8	0.00
6	–	–	11	12	■■97.88	7.9	11.7	■■86.55

Point no.	dx	dy	LAS			Danish		
			dx	Dy	T_j	dx	dy	T_j
(b)								
1	–	–	2.6	3.5	0.01	3.6	3.5	0.01
2	–	–	2.4	3.2	0.01	4.5	3.3	0.01
3	–	–	2.1	3.3	1.44	4.3	4.3	1.54
4	–	–	4.5	5.1	■■12.16	4.7	5.3	2.23
5	–	–	2.4	1.4	0.00	3.5	1.4	0.00
6	–	–	5.3	5.2	■■91.81	5.4	6.3	■■92.00

Table 4 The simulated rigid body and single point displacements in southern block of the network (dimension in mm)

Point no.	Simulated block shift		Estimated displacements					
	dx	dy	LS method			Robust M-estimators		
			dx	dy	T_j	dx	dy	T_j
(a)								
7	5	5	11	14	■■38.76	6.6	7.7	0.05
8	5	5	12	14	■■13.27	5.9	9.7	■■25.44
9	5	5	14	10	■■40.74	4.6	7.6	0.99
10	5	5	10	14	■7.06	0.5	9.6	0.03
11	5	5	13	14	4.86	3.7	9.6	0.45
12	5	5	12	14	■■28.58	6.6	7.8	■■54.45

Point no.	dx	dy	LAS			Danish		
			dx	dy	T_j	dx	dy	T_j
(b)								
7	5	5	6.1	7.3	0.00	6.2	7.4	0.02
8	5	5	5.3	9.2	■■21.37	5.4	9.3	■■22.23
9	5	5	4.2	5.1	0.82	4.3	5.3	0.90
10	5	5	4.2	6.2	0.01	4.4	6.4	0.02
11	5	5	3.2	6.1	0.11	3.3	6.3	0.14
12	5	5	4.9	6.3	■■50.34	4.1	6.4	■■49.99

point displacements are of primary interest. Then, robust estimation methods should be taken into consideration as an alternative to the conventional least squares method.

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